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Research Article

Kamenev-Type Oscillation Criteria for the Second-Order Nonlinear Dynamic Equations with Damping on Time Scales

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The oscillation of solutions of the second-order nonlinear dynamic equation $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)(x^{\Delta}(t))^{\gamma} + f(t, x(g(t))) = 0$, with damping on an arbitrary time scale $\mathbb{T}$, is investigated. The generalized Riccati transformation is applied for the study of the Kamenev-type oscillation criteria for this nonlinear dynamic equation. Several new sufficient conditions for oscillatory solutions of this equation are obtained.

1. Introduction

Much recent attention has been given to dynamic equations on time scales, or measure chains, and we refer the reader to the landmark paper of Hilger [1] for a comprehensive treatment of the subject. Since then, several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [2]. A book on the subject of time scales by Bohner and Peterson [3] also summarizes and organizes much of the time scale calculus.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The forward and the backward jump operators on any time scale $\mathbb{T}$ are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left scattered if $\rho(t) < t$, and right scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. For a function $f : \mathbb{T} \to \mathbb{R}$ the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}, \tag{1.1}$$

if f is continuous at t and t is right scattered. If t is not right scattered, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t^{+}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t^{+}} \frac{f(t) - f(s)}{t - s}, \tag{1.2}$$

provided this limit exists. A function $f : [a, b] \to \mathbb{R}$ is said to be right-dense continuous if it is right continuous at each right-dense point and there exists a finite left limit at all left-dense points, and f is said to be differentiable if its derivative exists. A useful formula dealing with the time scale is that

$$f^{\sigma} = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{1.3}$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$) of two differentiable functions f and g:

$$\begin{gathered} (fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma}, \\ \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}. \end{gathered} \tag{1.4}$$

The integration by parts formula is

$$\int_a^b f^{\Delta}(t) g(t) \Delta t = f(b) g(b) - f(a) g(a) - \int_a^b f^{\sigma}(t) g^{\Delta}(t) \Delta(t). \tag{1.5}$$

The function $f : \mathbb{T} \to \mathbb{R}$ is called *rd*-continuous if it is continuous at the right-dense points and if the left-sided limits exist in left-dense points. We denote the set of all $f : \mathbb{T} \to \mathbb{R}$ which are *rd*-continuous and regressive by $\Re$. If $p \in \Re$, then we can define the exponential function by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(t)}(p(\tau(t))) \Delta \tau\right) \tag{1.6}$$

for $t \in \mathbb{T}$, $s \in \mathbb{T}^k$, where $\xi_h(z)$ is the cylinder transformation, which is defined by

$$\xi_h(z) = \begin{cases} \dfrac{\log(1 + hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases} \tag{1.7}$$

Alternately, for $p \in \Re$ one can define the exponential function $e_p(\cdot, t_0)$, to be the unique solution of the IVP $x^{\Delta}(t) = p(t) x(t)$ with $x(t_0) = 1$.

The various-type oscillation and nonoscillation criteria for solutions of ordinary and partial differential equations have been studied extensively in a large cycle of works (see [4–31]).

In [27], the authors have considered second-order nonlinear neutral dynamic equation

$$\left(r(t)\left((y(t)+p(t)y(t-\tau))^{\Delta}\right)^{\gamma}\right)^{\Delta}+f(t,y(t-\delta))=0 \tag{1.8}$$

on a time scale $\mathbb{T}$. They have assumed that $\gamma > 0$ is a quotient of odd positive integers, τ and δ positive constants such that the delay functions $\tau(t) = t-\tau < t$ and $\delta(t) = t-\delta < t$ satisfy $\tau(t)$ and $\delta(t) : \mathbb{T} \to \mathbb{T}$ for all $t \in \mathbb{T}$, $r(t)$ and $p(t)$ real-valued positive functions defined on $\mathbb{T}$ and also they have supposed that

(H1) $\int_{t_0}^{\infty}(1/r(t))^{1/\gamma}\Delta t = \infty$, $0 \le p(t) < 1$,

(H2) $f(t,u) : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ is continuous such that $uf(t,u) > 0$ for all $u \neq 0$ and there exists a nonnegative function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)| \ge q(t)|u^{\gamma}|$

and were concerned with oscillation properties of (1.8). In [28], Saker has considered second-order nonlinear neutral delay dynamic equation

$$\left(r(t)\left((y(t)+p(t)y(t-\tau))^{\Delta}\right)^{\gamma}\right)^{\Delta}+f(t,y(t-\delta))=0, \tag{1.9}$$

when $\gamma \ge 1$ is an odd positive integer with $r(t)$ and $p(t)$ real-valued positive functions defined on $\mathbb{T}$. The author also has improved some well-known oscillation results for second-order neutral delay difference equations. Agarwal et al. [29] have considered the second-order perturbed dynamic equation

$$\left(r(t)\left(x^{\Delta}\right)^{\gamma}\right)^{\Delta}+F(t,x(t))=G\left(t,x(t),x^{\Delta}(t)\right), \tag{1.10}$$

where $\gamma \in \mathbb{N}$ is odd and they have interested in asymptotic behavior of solutions of (1.10). Saker et al. [30] have studied the second-order damped dynamic equation with damping

$$\left(a(t)x^{\Delta}(t)\right)^{\Delta}+p(t)x^{\Delta^{\sigma}}(t)+q(t)(fox^{\sigma})=0, \tag{1.11}$$

when $a(t)$, $p(t)$, and $q(t)$ are positive real-valued rd-continuous functions and they have proved that if $\int_{t_0}^{\infty}(e_{-p/r}(t,t_0)/r(t))\Delta t = \infty$ and $\int_{t_0}^{\infty}(e_{-p/r}(t,t_0)/r(t))\Delta t < \infty$, then every solution of (1.11) is oscillatory.

In the present paper, we consider the second order nonlinear dynamic equation

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+p(t)\left(x^{\Delta}(t)\right)^{\gamma}+f(t,x(g(t)))=0, \tag{1.12}$$

where p, r are real-valued, nonnegative, and right-dense continuous function on a time scale $\mathbb{T} \subset \mathbb{R}$, with $\sup\mathbb{T} = \infty$ and γ is a quotient of odd positive integers. We assume that $g : \mathbb{T} \to \mathbb{T}$ is a nondecreasing function and such that $g(t) \ge t$, for $t \in \mathbb{T}$ and $\lim_{t\to\infty}g(t) = \infty$. The function $f \in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$ is assumed to satisfy $uf(t,u) > 0$, for $u \neq 0$ and there exists a positive

rd-continuous function q defined on $\mathbb{T}$ such that $|f(t,u)/u^{\gamma}| \geq q(t)$ for $u \neq 0$. Throughout this paper we assume that

$$\int_{t_0}^{\infty} \left(\frac{e_{-p/r}(t,t_0)}{r(t)} \right)^{1/\gamma} \Delta t = \infty. \tag{A^*}$$

Since we are interested in the oscillatory of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$ and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$. The oscillation of solutions of the second-order nonlinear dynamic equation (1.12) with damping on an arbitrary time scale $\mathbb{T}$ is investigated. The generalized Riccati transformation is applied for the study of the Kamenev-type oscillation criteria for this nonlinear dynamic differential equation. Several new sufficient conditions for oscillatory solutions of this equation are obtained.

A solution $x(t)$ of (1.12) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory.

2. Preliminary Results

Lemma 2.1. *Assume that the condition* (A^*) *is satisfied and* (1.12) *has a positive solution* $x(t)$ *on* $[t_0,\infty)_{\mathbb{T}}$. *Then there exists a sufficiently large* $t_1 \in [t_0,\infty)_{\mathbb{T}}$ *such that*

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < 0, x^{\Delta}(t) > 0 \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}}. \tag{2.1}$$

Proof. Let $t_1 \in [t_0,\infty)$ such that $x(g(t)) > 0$ on $[t_1,\infty)$. Since $x(t)$ is positive nonoscillatory solution of (1.12) we can assume that $x^{\Delta}(t) < 0$ for all large t. Then without loss of generality we take $x^{\Delta}(t) < 0$ for all $t \geq t_2 \geq t_1$. From (1.12) it follows that

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + p(t)\left(x^{\Delta}(t)\right)^{\gamma} = -f(t,x(g(t))) < 0 \tag{2.2}$$

and so

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + p(t)\left(x^{\Delta}(t)\right)^{\gamma} < 0. \tag{2.3}$$

Define $y(t) = -r(t)(x^{\Delta}(t))^{\gamma}$. Hence

$$y^{\Delta}(t) + \frac{p(t)}{r(t)} y(t) > 0, \tag{2.4}$$

and it implies that

$$y(t) > y(t_2) e_{-p/r}(\cdot,t_2). \tag{2.5}$$

Then

$$-r(t)\left(x^{\Delta}(t)\right)^{\gamma} > -r(t_2)\left(x^{\Delta}(t_2)\right)^{\gamma} e_{-p/r}(\cdot, t_2), \tag{2.6}$$

and therefore

$$x^{\Delta}(t) \leq r^{1/\gamma}(t_2)\left(x^{\Delta}(t_2)\right)\left(\frac{e_{-p/r}(\cdot, t_2)}{r(t)}\right)^{1/\gamma}. \tag{2.7}$$

Next an integration for $t > t_3 \geq t_2$ and by (A^*) gives

$$x(t) \leq x(t_3) + r^{1/\gamma}(t_2)\left(x^{\Delta}(t_2)\right)\int_{t_3}^{t}\left(\frac{e_{-p/r}(s, t_2)}{r(s)}\right)^{1/\gamma}\Delta s \longrightarrow -\infty \quad \text{as } t \longrightarrow \infty \tag{2.8}$$

which is a contradiction. Hence $x^{\Delta}(t)$ is not negative for all large t and so $x^{\Delta}(t) > 0$ for all $t \geq t_1$. This completes the proof of Lemma 2.1. □

We now define

$$\begin{aligned}
\alpha_1(t) &:= \left(\frac{1}{r(t)}\int_t^{\infty} q(s)\Delta s\right)^{(1-\gamma)/\gamma} \\
\alpha_2(t,u) &:= \left(r^{1/\gamma}(t)\int_u^t \frac{\Delta s}{r^{1/\gamma}(s)}\right)^{\gamma-1} \\
\alpha(t) &:= \begin{cases} \alpha_1(t), & 0 < \gamma \leq 1, \\ \alpha_2(t, t_1), & \gamma \geq 1. \end{cases}
\end{aligned} \tag{2.9}$$

Lemma 2.2. *Assume that* (A^*) *holds and* (1.12) *has a positive solution* $x(t)$ *on* $[t_0, \infty)_{\mathbb{T}}$*. Then there exists a sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *such that if* $0 < \gamma \leq 1$ *for* $t \geq t_1$ *one has*

$$\left(\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\right)^{1-\gamma} \geq \alpha_1(t). \tag{2.10}$$

Whereas, if $\gamma \geq 1$*, one has*

$$\left(\frac{x(t)}{x^{\Delta}(t)}\right)^{\gamma-1} \geq \alpha_2(t, t_1) \quad \text{for } t \geq t_1. \tag{2.11}$$

Proof. As in the proof of Lemma 2.1, there is a sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$x(t) > 0, \quad x^{\Delta}(t) > 0, \quad \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < 0, \quad \text{for } t \geq t_1. \tag{2.12}$$

From (1.12) and (2.12) it follows that

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+p(t)\left(x^{\Delta}(t)\right)^{\gamma}=-f(t,x(g(t)))<0, \tag{2.13}$$

and so

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}<-f(t,x(g(t))). \tag{2.14}$$

Then

$$\begin{aligned} r(t)\left(x^{\Delta}(t)\right)^{\gamma} &\geq \int_t^{\infty} f(s,x(g(s)))\Delta s \geq \int_t^{\infty} q(s)x^{\gamma}(g(s))\Delta s \\ &\geq x^{\gamma}(g(t))\int_t^{\infty} q(s)\Delta s \geq (x^{\sigma}(t))^{\gamma}\int_t^{\infty} q(s)\Delta s. \end{aligned} \tag{2.15}$$

Next, when $0<\gamma\leq 1$, we get

$$\left(\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\right)^{1-\gamma}\geq \alpha_1(t) \quad \text{for } t\geq t_1. \tag{2.16}$$

Finally, since $r(t)(x^{\Delta}(t))^{\gamma}$ is decreasing on $[t_1,\infty)_{\mathbb{T}}$ for $\gamma\geq 1$, we get

$$\begin{aligned} x(t)\geq x(t)-x(t_1) &= \int_{t_1}^{t}\frac{\left(r(s)(x^{\Delta}(s))^{\gamma}\right)^{1/\gamma}}{r^{1/\gamma}(s)}\Delta s \\ &\geq \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{1/\gamma}\int_{t_1}^{t}\frac{1}{r^{1/\gamma}(s)}\Delta s, \end{aligned} \tag{2.17}$$

and we obtain

$$\left(\frac{x(t)}{x^{\Delta}(t)}\right)^{\gamma-1}\geq \alpha_2(t,t_1) \quad \text{for } t\geq t_1. \tag{2.18}$$

□

3. Main Results

Theorem 3.1. *Assume that (A^*) holds and there exist a function $\phi(t)$ such that $r(t)\phi(t)$ is a Δ-differentiable function and a positive real rd-functions Δ-differentiable function $z(t)$ such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\Psi(s)-\frac{1}{4}\frac{r(s)(\nu(s))^2}{\gamma z(s)\alpha(s)}\right]\Delta s=\infty, \tag{3.1}$$

where

$$\Psi(t) = -z(t)\left(q(s) - \left(r(t)\phi(t)\right)^{\Delta} + \frac{\gamma\alpha(t)}{r(t)}\left(p(t)\left(r(t)\phi(t)\right)^{\sigma} + \left(\left(r(t)\phi(t)\right)^{\sigma}\right)^{2}\right)\right),$$
$$\nu(t) = z^{\Delta}(t) - \frac{\gamma z(t)\alpha(t)}{r(t)}\left(p(t) - 2\left(r(t)\phi(t)\right)^{\sigma}\right). \tag{3.2}$$

Then every solution of (1.12) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.12). Without loss of generality, there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, so that $x(t)$ satisfies the conclusions of Lemmas 2.1 and 2.2 on $[t_0, \infty)_{\mathbb{T}}$. Define the function $w(t)$ by Riccati substitution

$$w(t) = z(t)r(t)\left(\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma} + \phi(t)\right), \quad t \geq t_1. \tag{3.3}$$

Then $w(t)$ satisfies

$$\begin{aligned} w^{\Delta}(t) &= \left(\frac{z(t)}{x^{\gamma}(t)}\right)\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \left(\frac{z(t)}{x^{\gamma}(t)}\right)^{\Delta}\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma} \\ &\quad + z(t)\left(r(t)\phi(t)\right)^{\Delta} + z^{\Delta}(t)\left(r(t)\phi(t)\right)^{\sigma}, \\ w^{\Delta}(t) &= \left(\frac{z(t)}{x^{\gamma}(t)}\right)\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \left(\frac{z^{\Delta}(t)x^{\gamma}(t) - z(t)(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)(x^{\gamma}(t))^{\sigma}}\right)\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma} \\ &\quad + z(t)\left(r(t)\phi(t)\right)^{\Delta} + z^{\Delta}(t)\left(r(t)\phi(t)\right)^{\sigma}. \end{aligned} \tag{3.4}$$

From (1.12) and the definition of $w(t)$ for $t \geq t_1$ it follows that

$$\begin{aligned} w^{\Delta}(t) &= \left(\frac{z(t)}{x^{\gamma}(t)}\right)\left(-p(t)\left(x^{\Delta}(t)\right)^{\gamma} - f\left(t, x\left(g(t)\right)\right)\right) + z^{\Delta}(t)\frac{\left(r(t)(x^{\Delta}(t))^{\gamma}\right)^{\sigma}}{(x^{\gamma}(t))^{\sigma}} \\ &\quad - z(t)\frac{(x^{\gamma}(t))^{\Delta}\left(r(t)(x^{\Delta}(t))^{\gamma}\right)^{\sigma}}{x^{\gamma}(t)(x^{\gamma}(t))^{\sigma}} + z(t)\left(r(t)\phi(t)\right)^{\Delta} + z^{\Delta}(t)\left(r(t)\phi(t)\right)^{\sigma}. \end{aligned} \tag{3.5}$$

Using the fact that $f(t, x(g(t))) \geq q(t)x^{\gamma}(g(t))$ and $x(t)$ is a increasing function, we obtain

$$\begin{aligned} w^{\Delta}(t) &\leq -z(t)q(t) - z(t)p(t)\frac{(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)} + z^{\Delta}(t)\left(\left(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\right)^{\sigma} + \left(r(t)\phi(t)\right)^{\sigma}\right) \\ &\quad - z(t)\frac{(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \left(r(t)\phi(t)\right)^{\sigma}\right) + z(t)\left(r(t)\phi(t)\right)^{\Delta}. \end{aligned} \tag{3.6}$$

Now we consider the following two cases: $0 < \gamma \leq 1$ and $\gamma > 1$.

In the first case $0 < \gamma \le 1$. Using the Pötzsche chain rule (see, [3]), we obtain

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 \left[x(t) + h\mu(t)x^{\Delta}(t)\right]^{\gamma-1} dh\, x^{\Delta}(t) \ge \gamma (x^{\sigma}(t))^{\gamma-1} x^{\Delta}(t). \tag{3.7}$$

Using (3.7) in (3.6) for $t \ge t_1$, we get

$$\begin{aligned} w^{\Delta}(t) \le{}& -z(t)q(t) - \gamma z(t)p(t)\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\left(\frac{x^{\sigma}(t)}{x(t)}\right)^{\gamma} + z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} \\ & - \gamma z(t)\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\left(\frac{x^{\sigma}(t)}{x(t)}\right)^{\gamma}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - (r(t)\phi(t))^{\sigma}\right) + z(t)(r(t)\phi(t))^{\Delta}. \end{aligned} \tag{3.8}$$

By Lemmas 2.1 and 2.2, for $t \ge t_1$, we have that

$$\begin{gathered} \frac{x^{\Delta}(t)}{x^{\sigma}(t)} = \frac{1}{r(t)}\frac{r(t)(x^{\Delta}(t))^{\gamma}}{(x^{\gamma}(t))^{\sigma}}\left(\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\right)^{1-\gamma} \ge \frac{\alpha_1(t)}{r(t)}\frac{\left(r(t)(x^{\Delta}(t))^{\gamma}\right)^{\sigma}}{(x^{\gamma}(t))^{\sigma}}, \\ \frac{x^{\sigma}(t)}{x(t)} \ge 1. \end{gathered} \tag{3.9}$$

In the view of (3.8), and (3.9) we get

$$\begin{aligned} w^{\Delta}(t) \le{}& -z(t)q(t) + z(t)(r(t)\phi(t))^{\Delta} - \gamma z(t)p(t)\frac{\alpha_1(t)}{r(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - (r(t)\phi(t))^{\sigma}\right) \\ & + z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{\alpha_1(t)}{r(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - (r(t)\phi(t))^{\sigma}\right)^{2}. \end{aligned} \tag{3.10}$$

In the second case $\gamma > 1$. Applying the Pötzsche chain rule (see, [3]), we obtain

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 \left[x(t) + h\mu(t)x^{\Delta}(t)\right]^{\gamma-1} dh\, x^{\Delta}(t) \ge \gamma (x(t))^{\gamma-1} x^{\Delta}(t). \tag{3.11}$$

In the view of (3.11), (3.6) yields

$$\begin{aligned} w^{\Delta}(t) \le{}& -z(t)q(t) + z(t)(r(t)\phi(t))^{\Delta} - \gamma z(t)p(t)\frac{(x(t))^{\gamma-1}}{x^{\gamma}(t)}x^{\Delta}(t) \\ & + z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{(x(t))^{\gamma-1}}{x^{\gamma}(t)}x^{\Delta}(t)\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - (r(t)\phi(t))^{\sigma}\right). \end{aligned} \tag{3.12}$$

By Lemmas 2.1 and 2.2, we have that

$$\frac{x^{\Delta}(t)}{x(t)} = \frac{1}{r(t)}\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\left(\frac{x(t)}{x^{\Delta}(t)}\right)^{\gamma-1} \ge \frac{\alpha_2(t,t_1)}{r(t)}\frac{\left(r(t)(x^{\Delta}(t))^{\gamma}\right)^{\sigma}}{(x^{\gamma}(t))^{\sigma}}. \tag{3.13}$$

By (3.13), (3.12), and then using the definition of $w(t)$, we get

$$\begin{aligned} w^{\Delta}(t) \le& -z(t)q(t) + z(t)\left(r(t)\phi(t)\right)^{\Delta} - \gamma z(t)p(t)\frac{\alpha_2(t,t_1)}{r(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \left(r(t)\phi(t)\right)^{\sigma}\right) \\ &+ z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{\alpha_2(t,t_1)}{r(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \left(r(t)\phi(t)\right)^{\sigma}\right)^2. \end{aligned} \tag{3.14}$$

Using (3.10), (3.14), and the definitions of $\Psi(t), \nu(t)$, and $\alpha(t)$ for $\gamma > 0$, we get

$$w^{\Delta}(t) \le -\Psi(t) + \nu(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{\alpha(t)}{r(t)}\frac{\left(w^{\sigma}(t)\right)^2}{\left(z^{\sigma}(t)\right)^2}. \tag{3.15}$$

Then, we can write

$$w^{\Delta}(t) \le -\Psi(t) + \frac{r(t)(\nu(t))^2}{4\gamma z(t)\alpha(t)} - \left[\sqrt{\frac{\gamma z(t)\alpha(t)}{r(t)}}\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \frac{1}{2}\sqrt{\frac{r(t)}{\gamma z(t)\alpha(t)}}\nu(t)\right]^2, \tag{3.16}$$

and so, we get

$$w^{\Delta}(t) \le -\left[\Psi(t) - \frac{r(t)(\nu(t))^2}{4\gamma z(t)\alpha(t)}\right]. \tag{3.17}$$

Integrating (3.17) with respect to s from t_1 to t, we get

$$w(t) - w(t_1) \le -\int_{t_1}^{t}\left[\Psi(s) - \frac{r(s)(\nu(s))^2}{4\gamma z(s)\alpha(s)}\right]\Delta s, \tag{3.18}$$

and this implies that

$$\int_{t_1}^{t}\left[\Psi(s) - \frac{r(s)(\nu(s))^2}{4\gamma z(s)\alpha(s)}\right]\Delta s \le |w(t_1)| \tag{3.19}$$

which contradicts to assumption (3.1). This completes the proof of Theorem 3.1. □

Corollary 3.2. *Assume that* (A^*) *holds. If*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[q(s) + \frac{\gamma\alpha(s)p^2(s)}{4r(s)}\right]\Delta s = \infty, \tag{3.20}$$

then every solution of (1.12) *is oscillatory.*

Example 3.3. Consider the nonlinear dynamic equation

$$\left(t^{-\gamma}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\frac{1}{2}t^{-1-\gamma}\left(x^{\Delta}(t)\right)^{\gamma}+\frac{1}{t^{1/\gamma}}x^{\gamma}(g(t))=0,\quad t\in[t_0,\infty)_{\mathbb{T}},\ \mathbb{T}=2^{\mathbb{N}},\tag{3.21}$$

where $\gamma\geq 1$ is the quotient of the odd positive integers. We have that $p(t)=(1/2)(t^{-1-\gamma})$, $q(t)=1/t^{1/\gamma}$ and $r(t)=t^{-\gamma}$. If $\mathbb{T}=2^{\mathbb{N}}$, then $\sigma(t)=2t$ and $e_{-1/\sigma(t)}(t,t_0)=t_0/t$. So we get $e_{-p/r}(t,t_0)=t_0/t$. It is clear that (A^*) holds. Indeed,

$$\begin{gathered}\int_{t_0}^{t}\left(\frac{e_{-p/r}(\cdot,t_0)}{r(s)}\right)^{1/\gamma}\Delta s=(t_0)^{1/\gamma}\int_{t_0}^{t}\frac{1}{s^{(1/\gamma)-1}}\Delta s=\infty,\\ \alpha_2(t,t_0)=\left((r(t))^{1/\gamma}\int_{t_0}^{t}\frac{\Delta s}{(r(s))^{1/\gamma}}\right)^{\gamma-1}=t^{1/(\gamma-1)}\left(\int_{t_0}^{t}\frac{\Delta s}{s^{-1}}\right)^{\gamma-1},\end{gathered}\tag{3.22}$$

and then

$$\int_{t_0}^{t}\frac{\Delta s}{s^{-1}}=\infty\tag{3.23}$$

and so we can find $t_*\geq t_1$ such that $\int_{t_0}^{t}\Delta s/r^{1/\gamma}\geq 1$ for $t\geq t_*$. Then we can see from Corollary 3.2 that it follows that

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\frac{1}{s^{1/\gamma}}+\frac{\gamma\alpha(s)(p(s))^2}{4r(s)}\right]\Delta s=\infty,\tag{3.24}$$

and therefore every solution of (3.21) is oscillatory.

Now, let us introduce the class of functions $\Re$.

Let $\mathbb{D}_0\equiv\{(t,s)\in\mathbb{T}^2: t>s\geq t_0\}$ and $\mathbb{D}\equiv\{(t,s)\in\mathbb{T}^2: t\geq s\geq t_0\}$. The function $H\in C_{rd}(\mathbb{D},\mathbb{R})$ has the following properties:

$$H(t,t)=0,\quad t\geq t_0,\quad H(t,s)>0,\quad \text{on } \mathbb{D}_0,\tag{3.25}$$

and H has a continuous Δ-partial derivative $H_s^{\Delta}(t,s)$ on $\mathbb{D}_0$ with respect to the second variable. (H is rd-continuous function if H is rd-continuous function in t and s.)

Theorem 3.4. *Assume that the conditions of Lemma 2.1 are satisfied. Furthermore, suppose that there exist functions $H, H_s^{\Delta}\in C_{rd}(\mathbb{D},\mathbb{R})$ such that* (3.25) *holds and there exist a function $\phi(t)$ with $r(t)\phi(t)$ a Δ-differentiable function and a positive Δ-differentiable function $z(t)$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\Psi(s)-\frac{r(s)}{4\gamma H(t,s)z(s)\alpha(s)}\varphi^2(t,s)\right]\Delta s=\infty,\tag{3.26}$$

where $\varphi(t,s)=[H_s^{\Delta}(t,s)+H(t,s)\nu(s)]$. Then every solution of (1.12) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that (1.12) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then without loss of generality, there is a sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusions of Lemmas 2.1 and 2.2 on $[t_0,\infty)_{\mathbb{T}}$. Consider the generalized Riccati substitution

$$w(t) = z(t)r(t)\left(\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma} + \phi(t)\right). \tag{3.27}$$

We proceed as Theorem 3.1 and from (3.15) it follows that

$$w^{\Delta}(t) \le -\Psi(t) + \nu(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{\alpha(t)}{r(t)}\frac{(w^{\sigma}(t))^2}{(z^{\sigma}(t))^2}. \tag{3.28}$$

Multiplying both sides of (3.28) by $H(t,s)$ and integrating with respect to s from t_1 to t ($t \ge t_1$), we obtain

$$\begin{aligned}\int_{t_1}^{t} H(t,s)\Psi(s)\Delta(s) \le -\int_{t_1}^{t} H(t,s)w^{\Delta}(s) + \int_{t_1}^{t} H(t,s)\nu(s)\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\Delta s \\ - \int_{t_1}^{t} \gamma H(t,s)z(s)\frac{\alpha(s)}{r(s)}\frac{(w^{\sigma}(s))^2}{(z^{\sigma}(s))^2}\Delta s.\end{aligned} \tag{3.29}$$

Integrating by parts, we get

$$\begin{aligned}\int_{t_1}^{t} H(t,s)\Psi(s)\Delta(s) &\le H(t,t_1)w(t_1) + \int_{t_1}^{t} H_s^{\Delta}(t,s)w^{\sigma}(s)\Delta s + \int_{t_1}^{t} H(t,s)\nu(s)\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\Delta s \\ &\quad - \int_{t_1}^{t} \gamma H(t,s)z(s)\frac{\alpha(s)}{r(s)}\frac{(w^{\sigma}(s))^2}{(z^{\sigma}(s))^2}\Delta s, \\ \int_{t_1}^{t} H(t,s)\Psi(s)\Delta(s) &\le H(t,t_1)w(t_1) + \int_{t_1}^{t}\left[H_s^{\Delta}(t,s) + H(t,s)\nu(s)\right]\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\Delta s \\ &\quad - \int_{t_1}^{t} \gamma H(t,s)z(s)\frac{\alpha(s)}{r(s)}\frac{(w^{\sigma}(s))^2}{(z^{\sigma}(s))^2}\Delta s.\end{aligned} \tag{3.30}$$

It is easy to see that

$$\begin{aligned}\int_{t_1}^{t} H(t,s)\Psi(s)\Delta(s) \le H(t,t_1)w(t_1) + \int_{t_1}^{t} \varphi(t,s)\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\Delta s \\ - \int_{t_1}^{t} \gamma H(t,s)z(s)\frac{\alpha(s)}{r(s)}\frac{(w^{\sigma}(s))^2}{(z^{\sigma}(s))^2}\Delta s,\end{aligned} \tag{3.31}$$

where

$$\varphi(t,s) = \left[H_s^{\Delta}(t,s) + H(t,s)\nu(s)\right]. \tag{3.32}$$

Then we can write

$$\int_{t_1}^{t} H(t,s)\Psi(s)\Delta(s) \le H(t,t_1)w(t_1) + \int_{t_1}^{t} \frac{r(s)\varphi^2(t,s)}{4\gamma H(t,s)z(s)\alpha(s)}\Delta s$$
$$- \int_{t_1}^{t} \left[\sqrt{\frac{\gamma H(t,s)z(s)\alpha(s)}{r(s)}}\frac{w^\sigma(s)}{z^\sigma(s)} - \frac{1}{2}\sqrt{\frac{r(s)}{\gamma H(t,s)z(s)\alpha(s)}}\varphi(t,s)\right]^2 \Delta s. \tag{3.33}$$

Hence

$$\begin{gathered}\int_{t_1}^{t} H(t,s)\Psi(s) - \frac{r(s)\varphi^2(t,s)}{4\gamma H(t,s)z(s)\alpha(s)}\Delta s \le H(t,t_1)w(t_1)\\ \limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\Psi(s) - \frac{r(s)\varphi^2(t,s)}{4\gamma H(t,s)z(s)\alpha(s)}\right]\Delta s \le w(t_1)\end{gathered} \tag{3.34}$$

which contradicts with assumption (3.26). This completes the proof of Theorem 3.4. □

Corollary 3.5. *Assume that (A^*) holds. Furthermore, suppose that there exist functions H, H_s^Δ, and $h \in C_{rd}(\mathbb{D},\mathbb{R})$ such that (3.25) holds and there exist a function $\phi(t)$ such that $r(t)\phi(t)$ is a Δ-differentiable function and a positive Δ-differentiable function $z(t)$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\Psi(s) - \frac{h^2(s)(z^\sigma(s))^2 r(s)}{4\gamma z(s)\alpha(s)}\right]\Delta s = \infty, \tag{3.35}$$

where $\Psi(t)$ is as defined in Theorem 3.1 and $H_s^\Delta = -h(t,s)\sqrt{H(t,s)} - H(t,s)\nu(t)/z^\sigma(t)$. Then every solution of (1.12) *is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.*

Theorem 3.6. *Assume that (A^*) holds and there exists a Δ-differentiable positive function $z(t)$ such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[z(s)q(s) - \frac{r(s)\xi^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}z^\gamma(s)}\right]\Delta s = \infty, \tag{3.36}$$

where

$$\xi(t) = z^\Delta(t) - \frac{z(t)p(t)}{r(t)}. \tag{3.37}$$

Then every solution of (1.12) *is oscillatory.*

Proof. Suppose that (1.12) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then without loss of generality, there is a sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusions of Lemmas 2.1 and 2.2 on $[t_0,\infty)_{\mathbb{T}}$. Consider the generalized Riccati substitution

$$w(t) = z(t)r(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}. \tag{3.38}$$

From (3.6) it follows that

$$w^{\Delta}(t) \le -z(t)q(t) - z(t)p(t)\frac{(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)} + z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - z(t)\frac{(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\frac{w^{\sigma}(t)}{z^{\sigma}(t)}. \tag{3.39}$$

In the same manner as in the proof of Theorem 3.1, we get

$$(x^{\gamma}(t))^{\Delta} \ge \begin{cases} \gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}, & 0<\gamma\le 1 \\ \gamma(x(t))^{\gamma-1}x^{\Delta}, & \gamma>1. \end{cases} \tag{3.40}$$

If $0<\gamma\le 1$, then we have that

$$w^{\Delta}(t) \le -z(t)q(t) + \left[z^{\Delta}(t) - \frac{z(t)p(t)}{r(t)}\right]\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{(x^{\sigma}(t))^{\gamma}}{x^{\gamma}(t)}\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\frac{w^{\sigma}(t)}{z^{\sigma}(t)}, \tag{3.41}$$

whereas, if $\gamma>1$, we have that

$$w^{\Delta}(t) \le -z(t)q(t) + \left[z^{\Delta}(t) - \frac{z(t)p(t)}{r(t)}\right]\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{x^{\sigma}(t)}{x(t)}\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\frac{w^{\sigma}(t)}{z^{\sigma}(t)}. \tag{3.42}$$

Using the fact that $x(t)$ is increasing and $(r(t)(x^{\Delta}(t))^{\gamma}$ is decreasing on $[t_0,\infty)_{\mathbb{T}}$, we get

$$x^{\sigma}(t) \ge x(t), \qquad x^{\Delta}(t) \ge \left(\frac{r^{\sigma}(t)}{r(t)}\right)^{1/\gamma}\left(x^{\Delta}(t)\right)^{\sigma}. \tag{3.43}$$

Using (3.41), (3.42), and (3.43), we obtain

$$w^{\Delta}(t) \le -z(t)q(t) + \xi(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - z(t)\frac{\gamma}{r^{1/\gamma}(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)}\right)^{\lambda}, \tag{3.44}$$

where $\lambda = (\gamma+1)/\gamma$. Define $A>0$ and $B>0$ by

$$A^{\lambda} = \frac{\gamma z(t)(w^{\sigma}(t))^{\lambda}}{(z^{\sigma}(t))^{\lambda}r^{1/\gamma}(t)}, \qquad B^{\lambda-1} = \frac{r^{1/(\gamma+1)}(t)\xi(t)}{\lambda\gamma^{1/\lambda}z^{1/\lambda}(t)}. \tag{3.45}$$

Then using the inequality (see [32])

$$\lambda AB^{\lambda-1} - A^{\lambda} \leq (\lambda-1)B^{\lambda}, \tag{3.46}$$

we obtain

$$\xi(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - z(t)\frac{\gamma}{r^{1/\gamma}(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)}\right)^{\lambda} \leq \frac{r(t)\xi^{\gamma+1}(t)}{(\gamma+1)^{\gamma+1}z^{\gamma}(t)}. \tag{3.47}$$

From this last inequality and (3.44) it follows that

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[z(s)q(s) - \frac{r(s)\xi^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}z^{\gamma}(s)}\right]\Delta s \leq w(t_1) \tag{3.48}$$

which contradicts with the assumption (3.36). Theorem 3.6 is proved. □

Example 3.7. Consider the second-order equation

$$\left(t^{\gamma}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \frac{1}{t^2}\left(x^{\Delta}(t)\right)^{\gamma} + \frac{1}{t}x^{\gamma}(g(t)) = 0, \tag{3.49}$$

where $\gamma = 1/3 \leq 1,\ r(t) = t^{1/3}, q(t) = 1/t,\ t \geq t_0 = 2$. Then it follows that

$$e_{-p/r}(t,2) \geq 1 - \int_{2}^{t}\frac{p(s)}{r(s)}\Delta s = 1 - \int_{2}^{t}s^{-7/3}\Delta s > \frac{1}{2} \tag{3.50}$$

for $t \geq 2$, and so

$$\int_{2}^{t}\left(\frac{1}{r(s)}e_{-p/r}(s,2)\right)^{1/\gamma}\Delta s \geq \left(\frac{1}{2}\right)^{3}\int_{2}^{t}\frac{1}{s}\Delta s \longrightarrow \infty \text{ as } t \longrightarrow \infty. \tag{3.51}$$

Hence (A^*) is satisfied. Now let $z(t) = 1$ for $t \geq 2$. Then

$$\limsup_{t\to\infty}\int_{2}^{t}\left[z(s)q(s) - \frac{r(s)\xi^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}z^{\gamma}(s)}\right]\Delta s = \limsup_{t\to\infty}\int_{2}^{t}\left[\frac{1}{s} - \frac{s^{-25/9}}{(4/3)^{4/3}}\right]\Delta s = \infty, \tag{3.52}$$

and so (3.36) is satisfied as well. Hence by Theorem 3.6, we have that (3.49) is oscillatory.

Theorem 3.8. *Assume that the conditions of Lemma 2.1 hold. Furthermore, suppose that there exist functions $H, H_s^{\Delta} \in C_{rd}(\mathbb{D},\mathbb{R})$ such that* (3.25) *holds and there exists a positive real rd-functions Δ-differentiable function $z(t)$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)z(s)q(s) - \frac{C^{\gamma+1}(t,s)r(s)}{(\gamma+1)^{\gamma+1}z^{\gamma}(s)(H(t,s))^{\gamma}}\right]\Delta s = \infty, \tag{3.53}$$

where $C(t,s) = H_s^{\Delta} z^{\sigma}(s) + H(t,s)\xi(t)$ *and* $\xi(t) = z^{\Delta}(t) - z(t)(p(t)/r(t))$. *Then every solution of* (1.12) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that (1.12) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then without loss of generality, there is a sufficiently large $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusions of Lemmas 2.1 and 2.2 on $[t_0,\infty)_{\mathbb{T}}$. Consider the generalized Riccati substitution

$$w(t) = z(t)r(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}. \tag{3.54}$$

By Theorem 3.6 and inequality (3.44)

$$w^{\Delta}(t) \le -z(t)q(t) + \xi(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - z(t)\frac{\gamma}{r^{1/\gamma}(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)}\right)^{\lambda}, \tag{3.55}$$

where $\lambda = (\gamma+1)/\gamma$. Multiplying both sides of (3.55) with $H(t,s)$ and integrating with respect to s from t_1 to t $(t \ge t_1)$, we get

$$\begin{aligned}\int_{t_1}^{t} H(t,s)z(s)q(s)\Delta s \le -\int_{t_1}^{t} H(t,s)w^{\Delta}(s)\Delta(s) + \int_{t_1}^{t} H(t,s)\xi(s)\frac{w^{\sigma}(s)}{z^{\sigma}(s)} \\ - \int_{t_1}^{t} H(t,s)z(s)\frac{\gamma}{r^{1/\gamma}(s)}\left(\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\right)^{\lambda}\Delta s.\end{aligned} \tag{3.56}$$

Integrating by parts and using (3.25), we obtain

$$\int_{t_1}^{t} H(t,s)z(s)q(s)\Delta s \le H(t,t_1)w(t_1)\int_{t_1}^{t} C(t,s)\frac{w^{\sigma}(s)}{z^{\sigma}(s)} - \int_{t_1}^{t}\frac{\gamma H(t,s)z(s)}{r^{1/\gamma}(s)}\left(\frac{w^{\sigma}(s)}{z^{\sigma}(s)}\right)^{\lambda}\Delta s. \tag{3.57}$$

Define $A > 0$ and $B > 0$ by

$$A^{\lambda} = \frac{\gamma H(t,s)z(t)(w^{\sigma}(t))^{\lambda}}{(z^{\sigma}(t))^{\lambda}r^{1/\gamma}(t)}, \qquad B^{\lambda-1} = \frac{r^{1/(\gamma+1)}(t)C(t,s)}{\lambda(\gamma H(t,s)z(s))^{1/\lambda}}. \tag{3.58}$$

Using the inequality (see [32])

$$\lambda AB^{\lambda-1} - A^{\lambda} \le (\lambda-1)B^{\lambda}, \tag{3.59}$$

we get

$$C(t,s)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \frac{\gamma H(t,s)z(t)}{r^{1/\gamma}(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)}\right)^{\lambda} \le \frac{r(t)C^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}H^{\gamma}(t,s)z^{\gamma}(t)}. \tag{3.60}$$

From this last inequality and (3.55) it follows that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)z(s)q(s)-\frac{r(s)C^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}H^{\gamma}(t,s)z^{\gamma}(t)}\right]\Delta s\le w(t_1) \tag{3.61}$$

which contradicts with the assumption (3.53). This completes the proof of Theorem 3.8. □

Corollary 3.9. *Assume that all conditions of Lemma 2.1 hold. Furthermore, suppose that there exist functions H, H_s^{Δ}, and $h\in C_{rd}(\mathbb{D},\mathbb{R})$ such that* (3.25) *holds and there exists a positive Δ-differentiable function $z(t)$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)z(s)q(s)-\frac{(-h(t,s))^{\gamma+1}r(s)}{(\gamma+1)^{\gamma+1}z^{\gamma}(s)}\right]\Delta s=\infty, \tag{3.62}$$

where $H_s^{\Delta}+H(t,s)\xi(t)/z^{\sigma}(s)=-h(t,s)(H(t,s))^{\gamma/(\gamma+1)}/z^{\sigma}(t)$. Then every solution of (1.12) *is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.*

Example 3.10. Consider the second-order dynamic equation

$$\left(t^{\gamma}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\frac{1}{t^2}\left(x^{\Delta}(t)\right)^{\gamma}+\frac{1}{t}x^{\gamma}(g(t))=0, \tag{3.63}$$

where $t\in[t_0,\infty)_{\mathbb{T}}$, $t_1\ge t_0=2$, $\gamma=5/3\ge 1, q(t)=1/t$. It is easy to check that (A^*) holds. For $z(t)=1$ and $H(t,s)=(t-s)^2$, it immediately follows that

$$h(t,s)=\left\{(t-s)-(t-s)^2+(t-\sigma(s))\right\}(t-s)^{2\gamma/(\gamma+1)} \tag{3.64}$$

and so $-h(t,s)=0$. Hence,

$$\limsup_{t\to\infty}\frac{1}{H(t,2)}\int_{2}^{t}\left[H(t,s)z(s)q(s)-\frac{(-h(t,s))^{\gamma+1}r(s)}{(\gamma+1)^{\gamma+1}z^{\gamma}(s)}\right]\Delta s=\limsup_{t\to\infty}\frac{1}{t^2}\int_{2}^{t}\frac{1}{s}(t-s)^2\Delta s=\infty. \tag{3.65}$$

Therefore by Corollary 3.9, every solution of (3.63) is oscillatory.

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