ТЕОРЕТИЧЕСКАЯ
И МАТЕМАТИЧЕСКАЯ
ФИЗИКА
Том 105, № 3
декабрь, 1995

R.S. Rutman

ON PHYSICAL INTERPRETATIONS OF FRACTIONAL INTEGRATION AND DIFFERENTIATION

Is there a relation between fractional calculus and fractal geometry? Can a fractional order system be represented by a causal dynamical model? These are the questions recently debated in the scientific community.

The author intends to answer to these questions. In the first part of the paper, some recently suggested models are reviewed and no convincing evidence is found for any dynamical model of a fractional order system having been built with the help of fractals. Linear filters with constant lumped parameters have a very limited use as approximations of fractional order systems. The model suggested in the paper is a state-space representation with parameters as functions of the independent variable. Regularization of fractional differentiation is considered and asymptotic error estimates, as well as simulation results, are presented.

1. FRACTIONAL INTEGRATION MODELS, OLD AND NEW, CORRECT AND WRONG

Fractional (more exactly, non-integer) order calculus has been developed in parallel with the conventional, integer order calculus. It was brought into being by Abel, Riemann and Liouville and has benefited from contribution by many authors throughout the 19th and 20th centuries. From the very beginning, fractional integration found a physical interpretation (Abel's mechanical model). In our time and especially with the arrival of tomography, physical models of fractional integration have been discovered and studied in spectrography, stereology, geophysical explorations[1]. The operator of fractional integration J^ν (usually, but not necessarily, $0 < \nu < 1$) defined on a suitable function space is usually written in the Riemann–Liouville form

$$(J_a^\nu h)(x) := \frac{1}{\Gamma(\nu)} \int_a^x \frac{h(r)}{(x-r)^{1-\nu}} dr = f(x), \qquad a > 0, \tag{1}$$

with the gamma-function as a conventional scaling factor. Inversion of the integral equation (1), i.e., finding $h(x)$ from the given $f(x)$, is the mathematical substance of the inverse problems arising in the applied problems.

Another broad field of fractional calculus applications is presented by the processes of transfer, such as physical and chemical diffusion [2]. In that field, one deals usually with differential equations of a non-integer order, ordinary or partial.

Finally, a third approach to fractional order systems treats them as signal processors, where it is assumed that the causality property should be retained in the model. The causality condition, natural from the physical viewpoint, assures that before an input signal $h(t)$

is applied to the system, no reaction $f(t)$ to that input cat be developed by the system:

$$h(t) \equiv 0 \quad \text{for} \quad t \leq 0 \quad \text{implies that} \quad f(t) \equiv 0 \quad \text{for} \quad t \leq 0. \tag{2}$$

A causal model is especially important for the simulation of a dynamical system represented by ordinary differential equations. If a causal model for fractional order integrators is available, it can be employed in a closed-loop simulation circuit, utilizing a general use simulation software to obtain solutions both for linear and non-linear problems.

In the first group of problems, one rarely, if ever, is concerned with causality: the inversion is done off-line. In the second group, there is a certain if tacit understanding of the fact that the solution of the differential equation in question should be causal; for this aspect, see the works of Kempfle et al., such as [3], or Weber [4].

It should be also mentioned that although Abel's apparatus was developed for an arbitrary order of integration, all the classical models deal with $\nu = 1/2$. Lately, however, reports have appeared on physical processes where the fractional integration of orders other than $1/2$ have been observed, including a controversial "ultra-slow diffusion".

We will be concerned here with causal dynamical models of fractional integration / differentiation, and we will start with the models debated recently in the scientific community.

Following Nigmatullin [5], consider the causal convolution integral

$$f(x) = \int_0^x g(x-r)h(r)dr \tag{3}$$

which transforms the input signal $h(x)$ into the output signal $f(x)$ according to the memory function (impulse response) $g(x)$. If $g(x) = \{1 \text{ for } x \geq 0, 0 \text{ for } x < 0\}$ (step function), we have the conventional first-order integration, and if $g(x)$ is the Dirac delta-function $\delta(x)$, this transformation amounts to an identical reproduction of the input (the zero-order integration). It is logical to infer that the fractional integration of the order ν, $0 < \nu < 1$, will have the memory function interpolating, in a sense, between the δ-function and the step function.

Note that the representation of the memory function for the fractional integral known since the works of Riemann and Abel follows directly from the definition Eq. (1):

$$g(x) = \frac{1}{\Gamma(\nu)x^{1-\nu}}. \tag{4}$$

It does indeed interpolate between the δ-function and the step-function in the sense that it attributes a strong emphasis to the current value of the signal in the convolution mechanism (the weight $g(0) = \infty$), but unlike the δ-function, its memory is not limited to the current value and distributed over the whole support of $g(x)$, albeit not evenly.

Ignoring the representation by Eq. (4), Nigmatullin is asserting in [5] that this interpolation is represented by the Cantor fractal function, i.e., by the set of values 0 or 1 alternating an infinite number of times on the support $[0, x]$, with the relative weights of 0 or 1 controlled by the value of ν. This "function", "normed to one", i.e. divided by x, is means to be the memory function of fractional integration (for more formal definitions of the fractals used in this section, see Appendix 1).

The mathematical inconsistency of the construction in [5] has been shown in the letter to this journal [6]. There would have been no need to further mention the hypothesis by Nigmatullin were it not for the following.

The above attempt is yet another manifestation of the irresistible temptation to link the fractional calculus and the fractal geometry. These efforts, stimulated mainly by the linguistic proximity of the two terms and an enormous publicity around the fractals and chaos, started immediately after the chaos theory was acclaimed by B. Mandelbrot, and have never stopped since. The confidence in the existence of such a relation was asserted at the International Conference on Fractional Calculus (Tokyo, 1990). A leading expert on fractional calculus was approached by a firm with the request to find a link between the two fields; he found none [7]. Finally, the two subjects were merged into a single panel at the International Summer School "Fractional and Hyperbolic Geometries/Fractional and Fractal Derivatives in Engineering, Applied Physics and Economics" that was held in Bordeaux in July 1994, with A. Oustaloup and A. Le Méhauté as coorganizers. The program of the school contains an opening by Le Méhauté entitled "Historical Review: From Lobachevski to R.R. Nigmatullin: The Kazan way of thinking".

For the school in Bordeaux, the hypothesis of [5] has been modified: now [2], the Cantor function has the support on a constant interval $[0, T]$, with T exceeding the duration of the process, and the "norming to one" is performed by dividing by T. By this modification, the conspicuous conflict with the notion of convolution has been removed, and the reader is invited to believe in the final result which is claimed to be the same as in [5]. In its original version, the hypothesis by Nigmatullin had at least some semblance of a physical meaning: the division by x did provide for a discontinuity at $x = 0$, just as in the true memory function Eq. (4). Of course, in the other aspects the true memory function and what is claimed in [5] and [2] are very different, and anyway, the final formula by Nigmatullin can be obtained neither by the original way as in [5] nor with the modification as in [2], which was observed by some participants in Bordeaux [8].

For a more complete picture of the interaction of the two fields, the following should be added. In the works of Zähle et al. [9], an alternative approach to fractional calculus is being developed based on the Cesáro derivative. This is one of the fractional operators among several in use differing mainly in their domains, and it is linked with the Riemann–Liouville approach by the way of the Marchaud derivative [10]. To demonstrate the power of their method on a "very bad function", the authors evaluate the Hausdorff measure of the Cantor middle-third set. Of course, this has nothing to do with a dynamical model of fractional integration; the same can be said about the works of Tricot [11] whose goal is to establish a relation between the fractal dimension of the curve and the maximal order of its derivative.

Returning to the program of the Bordeaux summer school, a lion's share of the presentations were devoted to control systems and other dynamical systems developed by Oustaloup and described in his two books [12] and numerous conference presentations. There, the setting of the closed-loop dynamical system is considered, with the open-loop frequency response $\hat{g}(i\omega)$, the Fourier transform of the impulse response $g(x)$; x is the time variable and ω the frequency.

Oustaloup is making use of the fact that if in the middle-frequency band $\omega' < \omega < \omega''$, $\hat{g}(i\omega)$ can be approximated as

$$\hat{g}(i\omega) = \frac{\text{constant}}{(i\omega)^{\nu}} \tag{5}$$

with $\nu = 1\frac{1}{2}$, the closed-loop system will have very good performance characteristics. Moreover, if the range $[\omega', \omega'']$ is wide enough, the system also exhibits a high robustness, or a low sensitivity to parameter changes. (If one defines the crossover frequency ω_0 by means of $|\hat{g}(i\omega_0)| = 1$, the "middle-frequency band" is determined by the requirement that the logarithm of the crossover frequency lie roughly in the middle of the interval $[\log\omega', \log\omega'']$.)

Although these facts have been known since the 1950's and recommendations of this type could be found in the standard textbooks on the control theory of that period, Oustaloup has found and developed a number of new interesting applications of these properties.

To implement the property Eq. (5), linear filters with lumped constant parameters are utilized in [12], with the transfer function $\hat{g}_{OU}(s)$ whose poles and zeros alternate and are placed at certain distances one from another within the range $[\log\omega', \log\omega'']$. Since Eq. (5) with constant $= 1$ represents the Fourier transform of $g(x)$ as in Eq. (4), these systems are claimed to be of the fractional order ν.

The fact remains that, contrary to the assertion, these systems are not fractional order systems. The device used in [12] can indeed provide for any, approximately constant, slope of the logarithm of the modulo of the frequency response in the middle-frequency band, just like $\hat{g}(i\omega)$ in Eq. (5). At the low frequencies and at $\omega = 0$, however, an arbitrary slope cannot be shaped by this device: it corresponds to the singularity of the transfer function $\hat{g}_{OU}(s)$ at $s = 0$; namely, it remains equal to the value $q = -$(multiplicity of the pole at the origin). Indeed, the well known necessary conditions of *finite realizabiliy* for linear time-invariant systems are

$$\hat{g}(i\omega) \sim c_1(i\omega)^q, \qquad \omega \longrightarrow 0, \tag{6}$$

$$\hat{g}(i\omega) \sim c_2(i\omega)^p, \qquad \omega \longrightarrow \infty, \tag{7}$$

with p and q integer (normally non-positive; non-positivity of p is required by another property of dynamical systems, *physical realizability*). Thus, a most important parameter of the system which is claimed to be represented by such a filter, namely the character of the singularity of $\hat{g}(s)$ at the origin, is distorted by such "approximation": for ν non-integer, it is a branching point where we should take the principal branch, and for ν integer, it is a pole. The component in the time response corresponding to this pole has a different asymptotic behavior than that for ν non-integer.

Suppose that in an appropriate normed space H an approximating model represented by the operator $M(U)$ is built retaining the property Eq. (6). Then it is impossible to assure that there exist a set of parameters U such that *for all* $\varepsilon > 0$ *and for all* $h \in H$, *there exists* $\delta(\varepsilon)$ *such that if* $\|h\| < \delta(\varepsilon)$, *then*

$$\left\| J^\nu h - M(U)h \right\| < \varepsilon \tag{8}$$

(*approximation property*). To illustrate this, it is sufficient to take $h(x)$ in Eq. (8) equal to the unit step. One can observe how the different singularities at the origin $\omega = 0$ create an unbounded discrepancy between the two systems when $x \to \infty$.

As long as the Oustaloup models are used strictly in the frequency domain in a limited band-pass where the transient response is of no concern, they are correct and may lead to interesting applications, as in electronic music synthesizers [13]. For the use in the time domain as "fractional order systems", they are manifestly inadequate.

Now consider a *state-space representation* for linear dynamical systems

$$\frac{dz}{dx} = Az(x) + Bh(x), \qquad f(x) = Cz(x) + Dh(x), \tag{9}$$

where $z(x)$ is the state vector and A, B, C and D are matrices of corresponding dimensions.

It is well known that any dynamical system of the form Eq. (9) with constant matrices has an input-output representation as a convolution integral Eq. (3), with

$$g(t) \longleftrightarrow \hat{g}(s) = C(sI - A)^{-1}B + D, \tag{10}$$

where the arrow denotes the Laplace transform and I is the identity matrix. The converse statement is, however, untrue, since the conditions of finite realizability Eqs. (6) and (7) impose restrictions. In particular, they prevent a finite realization of fractional order systems such as in Eq. (5) with ν non-integer. In other words, the character of the singularity of the integral Eq. (1) makes it impossible to find a representation of the type Eq. (9). A naive approximation such as in [12] has no chance to succeed.

2. REALIZATION OF FRACTIONAL INTEGRATION

We are going to look for the solution of the problem of physical realization, both for fractional integration and fractional differentiation, in the form which should be properly parametrized and causal.

Addressing first the fractional integration, we require that the condition Eq. (8) be met, which, since we are dealing with linear operators, is equivalent to

$$\text{for all} \quad h \in H, \quad \lim_{a \to 0} \left\| \left[M(U) - J^{\nu} \right] h \right\| = 0 \tag{11}$$

(it is understood here that the set U can be determined by a single parameter α).

A solution to the problem formulated above, i.e., a model of causal fractional integration, has been found in the form similar to Eq. (10) but with the matrices A and C *depending on the time variable* x:

$$\begin{aligned} \frac{dz(x)}{dx} &= A(x)z(x) + Bh(x), \\ f(x) &= C(x)z(x) \end{aligned} \tag{12}$$

($D = 0$). The dimensions of matrices are infinite, but after a proper parametrization, a model of a finite order n will represent the system, with the approximation condition Eq. (11) satisfied.

The determination of the matrices $A_{n \times n}(x)$, $B_{n \times 1}$, $C_{1 \times n}(x)$ is presented below briefly as a summary of operations to be performed. A proof can be found in Appendix 2.

The algorithm consists of the following steps:

A. Choice of the order of the system, n.

The order of the approximating system determines uniquely the set of the approximating parameters. Its choice is determined by a compromise between accuracy and complexity.

Some practical considerations are presented at the end of Sec. 3.

B. Determination of the zeros, poles and gain of the auxiliary transfer function

$$\hat{k}_{1-\nu,n}(s) = \frac{1}{s} g_{1-\nu,n} \prod_{k=2}^{n} \frac{s - \hat{\beta}_{k,1-n}}{s - \hat{\lambda}_{k,1-n}}. \tag{13}$$

This is done according to the formulas

$$\begin{aligned} \hat{\beta}_{k,1-\nu} &= -(k-2+\nu) \exp\left[\frac{1}{n+1}(k-2)^2 \right], \\ \hat{\lambda}_{k,1-\nu} &= -(k-1) \exp\left[\frac{1}{n+1}(1-k-\nu)^2 \right], \qquad k = 2, 3, \ldots, n, \\ g_{1-\nu,n} &= \frac{1}{\nu} \prod_{k=2}^{n} \frac{\hat{\lambda}_{k,1-\nu}}{\hat{\beta}_{k,1-\nu}}. \end{aligned} \tag{14}$$

C. Determination of the matrices A^*, B^* and C^*.
The formula

$$\hat{k}_{1-\nu,n}(s) = C^*(sI - A^*)^{-1}B^* \tag{15}$$

allows for an infinite number of the solutions A^*, B^* and C^*. A standard software, such as in MATLAB with Control Systems Toolbox, will return a set of the matrices satisfying Eq. (15), with an A diagonal.

D. Determination of the matrices A, B and C is performed according to the formulas

$$A(x) = \frac{A^*}{x}, \qquad B = \frac{B^*}{\Gamma(\nu)}, \qquad C(x) = x^{\nu-1}C^*. \tag{16}$$

An additional insight into the results is afforded by the analysis in the frequency domain. It can be shown that, by virtue of the transformation for the time variable $t \to e^t$ (see Appendix 2), the frequency variable ω has been subject to a transformation $\mathbf{\Omega} : \mathbf{R}^+ \to \mathbf{R}^+$. For $\nu = 1/2$,

$$\mathbf{\Omega} : \omega \longrightarrow \omega \tanh \pi\omega. \tag{17}$$

This "frequency warping" has altered the asymptotic behavior of the frequency response in the proximity of $\omega = 0$, so that

$$\left|\hat{k}_{1-\nu,n}(i\omega)\right| \sim \frac{1}{\omega} \quad \text{as} \quad \omega \longrightarrow 0, \tag{18}$$

i.e., the condition Eq. (6) has been met by the frequency response Eq. (18). As for the condition (7), it is met by the process of the parametrization. Thus, both conditions of finite realizability (6) and (7) are met by $\hat{k}_{1-\nu,n}(i\omega)$.

Some detail concerning the steps a and b can be found in the conference paper [14].

3. REALIZATION OF FRACTIONAL DIFFERENTIATION

Although historically the fractional derivative, following the classical pattern of the conventional (integer order) calculus, was introduced before the fractional integral, it should be emphasized that, as in the case of the conventional calculus, physical processes always have a character of integration (dissipation of energy and / or information), and the fractional derivative is best interpreted as the (left) inverse of the fractional integral Eq. (1). In keeping with this line of the argument, one should model the physical system in terms of integration (of fractional or integer order, for that matter) and then proceed with its inversion. This is well understood in all practical methods of tomography, be it Radonian (3-dimensional) or Abelian (2-dimensional). This may not be seen as explicitely in the transfer models involving differential equations (of fractional order), but the fact remains that the symbol of the derivative, by it of an integer or fractional order, is a tribute to the conventional mathematical notations tracing back to Leibniz. It is best interpreted physically as the cause of the effect represented by the integral. A recognition of this fact is reflected in the simulation techniques for the differential equations going back to Thomson (Kelvin) and using integrators but no differentiators. An attempt to reproduce in an exact manner a derivative of an integer or fractional order would violate the conditions of physical realizability referred to in connection with Eq. (7).

An inversion formula (Abel) is known for Eq. (1) for sufficiently well behaved functions $f(x)$:

$$(D_a^\nu f)(r) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dr} \int_a^r \frac{f(r)dx}{(r-x)^\nu} = h(r). \tag{19}$$

Inversion of the Riemann–Liouville integral is not a well-posed problem, which is reflected by the presence of differentiation in Eq. (19). Therefore, approximation by the way of parametrization should also play the role of *regularization* (or stabilization, stability being understood as the "bounded input–bounded output stability").

The algorithm of fractional differentiation of the order ν represented as a dynamical system in the state-variable representation

$$\frac{d}{dr}z(x) = P(x)z(x) + Q(x)f(x), \qquad h_n(x) = R(x)z(x) + S(x)f(x), \tag{20}$$

with x as the time variable, consists of the following steps (see a proof in Appendix 2 and some detail in [14]):

A. Choice of the order of the system, n.

Since the value of n is related (inversely) to the regularization constant

$$\alpha = \frac{1}{2-\hat{\lambda}_n}, \tag{21}$$

this choice is determined by conventional considerations for the selection of the value of the regularization parameter. Namely, in the absence of the information on the noise, a compromise between the approximation error and the complexity of the system is sought; and if some *a priori* information, quantitative or descriptive, on the signals and noise is available, one of the standard strategies, such as Bayesian, can be selected.

B. Determination of the zeros, poles and gain of the auxiliary transfer function $\hat{k}_{\nu,n}(s)$ is done in the same manner as for the fractional integration, Eqs. (13)–(14), with ν replaced by $1-\nu$.

C. Determination of the matrices P^*, Q^*, R^* and S^*.

The formula

$$s\hat{k}_{\nu,n}(s) = R^*(sI - P^*)^{-1}Q^* + S^* \tag{22}$$

has an infinite number of solutions for the set $\{P^*, Q^*, R^*, S^*\}$. All the comments in the corresponding section on fractional integration pertain here.

D. Determination of the matrices P, Q, R and S.

The formulas

$$\begin{aligned} P(x) &= \frac{P^*(x)}{x}, \qquad & Q(x) &= \frac{1}{\Gamma(1-\nu)}Q^*(x)\frac{1}{x}, \\ R(x) &= R^*(x)\frac{1}{x^\nu}, \qquad & S(r) &= \frac{1}{\Gamma(1-\nu)}S^*(x)\frac{1}{x^\nu} \end{aligned} \tag{23}$$

determine fully the causal dynamical system Eq. (20), with x as the "time" variable.

As opposed to Eq. (12), one can observe in Eq. (20) a term proportional to the input signal $f(x)$.

In Appendix 3, one can find a rigorous treatment of the regularization aspects of the method, including a stability proof based on the Marcinkievicz–Mikhlin theorem and asymptotic estimates of the regularization error.

A computer simulation of the dynamical subsystem [14] has demonstrated the technique to have a high degree of accuracy. Thus, with only seven factors in the product in Eq. (13), i.e., with $n = 8$, the root-mean-square error in simulations was of the order 10^{-4} both for fractional integration and differentiation, i.e. on a par with, or better than in, the comparable techniques (for $\nu = 1/2$) of Andersen; Bokasten; Maldonaldo et al.; Minerbo and Levy; Nestor and Olsen [15], under the same test signals. None of those techniques provides for an arbitrary ν nor for a causal solution.

4. CONCLUSIONS

1. A linear filter with constant parameters (an ordinary linear differential equation with constant parameters) cannot adequately represent a fractional order system.
2. No direct relation between fractional calculus and the fractals has been established as yet.
3. Fractional integral or fractional derivative can be represented as an infinite system of ordinary linear differential equations of the first order, with the coefficients depending on the independent variable.
4. If the independent variable is understood as the time, fractional integral or fractional derivative can be approximated by a dynamical system with time-dependent parameters. This system is linear, stable and causal, with the approximation error controlled by the order of the system.

Acknowledgements. The author gratefully acknowledges the support of the NATO Scientific Affairs Division by the way of the Grant CRG 940843 and of the Massachusetts Center for Marine Science and Technology. He is grateful to Université de Paris XI and Laboratoire des Signaux et Systémes for providing him with a financial and logistical support in a congenial atmosphere.

APPENDIX 1 (see e.g. [9])

Define the affine maps $A_1, A_2: \mathbf{R} \to \mathbf{R}$

$$A_1: x \longrightarrow \gamma x, \qquad A_2: x \longrightarrow 1-\gamma+\gamma x, \qquad 0<\gamma<1/2.$$

Next, define the sequence C_n of compact sets

$$C_0 := [0,1], \qquad C_n := A_1 C_{n-1} \cup A_2 C_{n-1}.$$

DEFINITION. The Cantor set

$$C = \bigcap_{n=0} C_n.$$

The middle-third Cantor set is C for $\gamma = 1/3$.

DEFINITION. The Cantor measure, or the devil's stairway is the Hausdorff measure on $\mathbf{R}$ restricted to C.

APPENDIX 2

A. Fractional integration. With the transformation of the independent variable

$$x = ae^t, \qquad r = ae^\tau \tag{A.1}$$

the integral Eq. (1) becomes

$$f(x) = \frac{x^{\nu-1}}{\Gamma(\nu)} \int_0^t \frac{h^t(r) r\, d\tau}{[1-e^{-(t-\tau)}]^{1-\nu}} = \frac{1}{\Gamma(\nu)} \left[h^t(t) a e^{t} * k_{1-\nu}(t)\right] x^{\nu-1}. \tag{A.2}$$

Here * denotes convolution,

$$h^t(\tau) = h(ae^\tau) \tag{A.3}$$

and

$$k_\nu(t) = (1-e^{-t})^{-\nu}. \tag{A.4}$$

The Laplace transform $\hat{k}_{1-\nu}(s)$ of $k_{1-\nu}(t)$ is the beta-function $B(s,\nu)$. Using the well-known relations for beta-functions, we get

$$\hat{k}_{1-\nu}(s) = \frac{1}{\nu s}\prod_{k=2}^{\infty}\frac{k}{k-1+\nu}\frac{s-\beta_{k,1-\nu}}{s-\lambda_{k,1-\nu}}, \tag{A.5}$$

where

$$\beta_{k,1-\nu} = -(k-2+\nu), \qquad \lambda_{k,1-\nu} = -(k-1). \tag{A.6}$$

One can proceed now with defining a finite-dimensional model by truncating the infinite product in Eq. (A.5). After the replacement of β_k and λ_k by $\hat{\beta}_k$ and $\hat{\lambda}_k$ as in Eq. (14), which improves drastically the rate of convergence [14], we arrive at Eq. (3).

In a state-space form, we have

$$\frac{dz^*}{dt} = A^*z^*(t) + B^*\varphi(t), \qquad \psi_n(t) = C^*z^*(t), \tag{A.7}$$

with the matrices $A^*_{n\times n}$, $B^*_{n\times 1}$ and $C^*_{1\times n}$ related to the transfer function $\hat{k}_{1-\nu,n}(s)$ by the equation

$$\hat{k}_{1-\nu,n}(s) = C^*(sI - A^*)^{-1}B^*. \tag{A.8}$$

In the model obtained, the input signal

$$\varphi(t) = \frac{1}{\Gamma(\nu)}h(ae^t)ae^t \tag{A.9}$$

passes the dynamical part represented by the state-space model Eq. (A.7), or the transfer function $\hat{k}_{1-\nu,n}(s)$. The resulting signal $\psi_n(t)$ is multiplied by $x^{\nu-1}$ producing the output

$$f_n(x) = \psi_n\left(\ln\frac{x}{a}\right)x^{\nu-1}. \tag{A.10}$$

The model defined by Eq. (A.7)–(A.10) is obviously non-causal: if simulation is required, the input signal $h(x)$ should be pre-recorded and turned into a function of t, the "model time variable" governing the processes in the dynamical subsystem Eq. (A.8).

Let us perform the substitution

$$t = \ln\frac{x}{a}, \qquad dt = \frac{dx}{x}. \tag{A.11}$$

It turns Eqs. (A.7) into Eqs. (12) where all the operations are performed "on line", i.e., in the "real time" variable x.

B. Fractional differentiation. A similar technique applied to Eq. (19) with the substitutions $x = ae^\tau$, $r = ae^t$, yields

$$h(r) = \frac{1}{\Gamma(1-\nu)}\left[f_t'(t) * k_\nu(t)\right]\frac{1}{r^\nu}, \tag{A.12}$$

with $f_t(t) = f(ae^t)$. After truncating the infinite product in Eq. (16), with $1-\nu$ replaced by ν, we obtain the transfer function

$$\hat{k}_{\nu,n}(s) = \frac{1}{(1-\nu)s}\prod_{k=2}^{n}\frac{k}{k-\nu}\frac{s+\beta_{k,\nu}}{s+\lambda_{k,\nu}} \tag{A.13}$$

with β_k and α_k as in Eq. (A.6) to be replaced by $\hat{\beta}_k$ and $\hat{\lambda}_k$ as in Eq. (14). Differentiation in Eq. (A.12) will result in cancelling the s in the transfer function, which yields the space-state representation

$$\frac{d}{dt}z^*(t) = P^*z^*(t) + Q^*\varphi(t), \qquad \psi_n(t) = R^*z^*(t) + S^*\varphi(t), \tag{A.14}$$

with the input and output signals defined by

$$\varphi(t) = \frac{1}{\Gamma(1-\nu)}f(ae^t), \qquad h_n(r) = \psi_n\left(\ln\frac{r}{a}\right)\frac{1}{r^\nu}. \tag{A.15}$$

Here the matrices $P^*_{n\times n}$, $Q^*_{n\times 1}$, $R^*_{1\times n}$ and $S^*_{1\times 1}$ obey the equation

$$\hat{k}_{\nu,n}(s) = R^*(sI - P^*)^{-1}Q^* + S^*. \tag{A.16}$$

The back substitution similar to Eq. (A.11) provides for the causal model Eqs. (20).

APPENDIX 3

We will keep the notation $\hat{\varphi}(i\omega)$ of the Fourier transform of the function $\varphi(t)$

$$\hat{\varphi}(i\omega) := \int_{-\infty}^{\infty}\varphi(t)e^{-i\omega t}dt \tag{A.17}$$

even if Fourier transform does not exist but $\varphi(t)$ such that $\varphi(t) = 0$ for $t < 0$ is Laplace-transformable, with a convergence abscissa $\sigma > 0$. In this case, the spectrum $\hat{\varphi}(i\omega)$ is defined as

$$\hat{\varphi}(i\omega) := \left[\int_0^{\infty}\varphi(t)e^{-st}dt,\ \mathrm{Re}\, s > \sigma\right]_{s=i\omega}. \tag{A.18}$$

Now, since the spectrum of the fractional differentiator $|\hat{g}(i\omega)|$ is unbounded as $\omega \to \infty$, the operator

$$B : f(t) \longrightarrow h(t) \tag{A.19}$$

is unbounded in the customary metric function spaces rendering the fractional differentiation ill-posed.

It will be shown here that $s\hat{k}_{\nu,n}(s)$, with $\hat{k}_{\nu,n}(s)$, as in Eq. (A.14), introduces the regularizer

$$B^\alpha : f(ae^t) \longrightarrow h^\alpha(t) \tag{A.20}$$

such that: 1) B_α is a bounded operator in the domain of B, and 2)

$$\lim_{\alpha\to 0}(B_\alpha f)\left(t = \ln\frac{x}{a}\right) = (Bf)(x). \tag{A.21}$$

We rewrite Eq. (A.12) as

$$h(ae^t) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\left(f(t)^*k_\nu(t)\right)(ae^t)^{-\nu} \tag{A.22}$$

and compare it with

$$h_n(ae^t) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\left(f(t)^*k_{\nu,n}(t)\right)(ae^t)^{-\nu}. \tag{A.23}$$

If we shall do the Fourier transform of the right-hand side of Eq. (A.23), the attenuation and differentiation will account for the following:

$$h_n(ae^t) \longleftrightarrow \frac{i\omega}{\Gamma(1-\nu)a^\nu}\hat{f}(i\omega+\nu)\hat{k}_{\nu,n}(i\omega+\nu). \tag{A.24}$$

It is easy to see that $\hat{k}_{\nu,n}(i\omega)$ as in Eq. (13), as well as $\left|\omega\frac{d}{d\omega}\left[i\omega\hat{k}_{\nu,n}(i\omega)\right]\right|$ are bounded for $-\infty < \omega < \infty$. Therefore, by the Marcinkiewicz–Mikhlin theorem on Fourier multipliers [13], the operator B_n^* is bounded in $L^p(\mathbf{R})$, $1 < p < \infty$.

It we let the regularization constant be

$$\alpha = (-\hat{\lambda}_n)^{-1} \tag{A.25}$$

and take into account the formulas Eqs. (A.12)–(A.17), the property $\lim\limits_{\alpha\to\infty} B^\alpha f = Bf$ has been obviously fulfilled. □

Now we turn to asymptotic estimates. Let

$$\hat{k}_{\nu,n}(s) = \hat{k}_\nu(s)\hat{e}_{\nu,n}(s). \tag{A.26}$$

Then, dropping the index ν,

$$\hat{h}_n(s) = \hat{h}(s)e_n(s). \tag{A.27}$$

Notice first that the function

$$\tilde{e}_n(s) = \frac{1}{(\alpha s)^\nu + 1} \tag{A.28}$$

with α as in Eq. (A.25) is asymptotically equivalent to $\hat{e}_n(s)$ as $\alpha \to 0$ and $|s| \to \infty$ or $|s| \to 0$. In the time domain,

$$\tilde{e}_n(t) = \alpha^\nu \nu t^{\nu-1} E_\nu'\left[(\hat{\lambda}_n t)^\nu\right], \tag{A.29}$$

where $E_\nu(z)$ is the Mittag–Leffler function and E_ν' its derivative. Replacing $\hat{e}_n(s)$ by $\tilde{e}_n(s)$, denote, for $t > 0$,

$$\varepsilon_n(t) :\sim \left\{h(t) - h(t)^*\tilde{e}_n(t)\right\}, \qquad \alpha \longrightarrow 0. \tag{A.30}$$

For $\nu = 1/2$, another asymptotically equivalent function

$$\bar{\varepsilon}_n(s) = \frac{1}{(\alpha s+1)^{1/2}} \longleftrightarrow \frac{1}{\sqrt{\pi\alpha t}}e^{-\frac{t}{\alpha}} \tag{A.31}$$

can be used. It is easy to show (for instance, by repeated integration by parts) that

$$\int_0^\infty h(t-\tau)\exp\left(-\frac{\tau}{\alpha}\right)(\pi\alpha\tau)^{-1/2}d\tau = h(t) - \frac{\alpha}{2}h'(t) + \frac{3}{8}\alpha^2 h''(t) + O(\alpha^3). \tag{A.32}$$

Thus

$$\varepsilon_\alpha(t) = \frac{\alpha}{2}h'(t) + O(\alpha^3), \tag{A.33}$$

i.e., the asymptotic error does not depend on $h(t)$ but rather on its derivative.

In the frequency domain, the relative error is

$$\bar{\varepsilon}_n(\omega) = \left(\left|\bar{e}_n(i\omega)\right| - \left|\hat{e}(i\omega)\right|\right) / \left|\hat{e}_n(i\omega)\right|, \tag{A.34}$$

where

$$\left|\bar{e}_n(i\omega)\right| = (\alpha^2\omega^2+1)^{-\nu/2}. \tag{A.35}$$

For $\nu = 1/2$,

$$\left|\hat{e}_n(i\omega)\right| = \prod_{k=1}^n \frac{k}{k-\frac{1}{2}}\left|\frac{\frac{1}{2}i\omega - \frac{1}{2} + (k-1)}{\frac{1}{2}i\omega + (k-1)}\right|\left(\frac{\pi}{2\omega}\coth\frac{\pi\omega}{2}\right)^{-\frac{1}{2}}, \tag{A.36}$$

and for any ω, the value $\left|\bar{\varepsilon}_n(\omega)\right|$ does not exceed ε_n/n, where $0,17 > \varepsilon_3 > \varepsilon_4 > \varepsilon_5 \ldots$.

References

[1] *Gorenflo R., Vessela S.* Abel integral equations: analysis and applications. Berlin: Springer-Verlag, 1991.

[2] *Nigmatullin R.R.* The temporal fractional integral and its physical sense. Géometrie fractale et hyperbolique / derivation fractionnaire et fractale: applications dans les science de l'ingenieur et en économie. Ecole d'été internationale. Document de Travail. Bordeaux (France), 3–4 Juillet 1994.

[3] *Beyer H., Kempfle S.* Dämpfungsbeschreibung mittels gebrochener Abteilungen. Zeinschrift für Angewandte Mathematik und Mechanik (in print).

[4] *Weber V.K.* // Research in integrodifferential equations in Kirgizia (in Russian). Frunze: Ilim, 1983. V. 16. P. 349–356; 1985. V. 18. P. 301–305.

[5] *Nigmatullin R.R.* // Theor. Mathem. Phys. 1992. V. 90. P. 242–257; Russian original TMF. 1992. V. 90. P. 354–368.

[6] *Rutman R.* // TMF. 1994. V. 100. P. 476–478.

[7] *Ross B., Rubin B.* (personal communications).

[8] *Kempfle S., Mainardi F., Nigmatullin R.R., Rubin B.* (personal communications).

[9] *Zähle M.* // Topology, measures, and fractals/Ed. C. Bandt et al. Berlin: Akademie Verlag 1992. V. 66. P. 121–126; *Patzschke N., Zähle M.* // Proc. Amer. Math. Soc. 1993. V. 117. P. 137–144.

[10] *Samko S.G., Kilbas A.A., Marichev O.I.* Integrals and derivatives of fractional order and some of their applications. London: Gordon and Breach, 1993 ; *Kiraykova V.* Generalized fractional calculus and applications. Harlow, Essex: Longman Sci. Tech., 1994.

[11] *Tricot C.* Dérivation fractionnaire et dimension fractale. Tech. Rep. 1532. CRM – Université de Montréal, 1988.

[12] *Oustaloup A.* Systéms asservis d'ordre fractionnaire. Paris: Masson, 1983 ; La commande CRONE. Paris: Hermes, 1991.

[13] *Matignon D.* Réprésentation en variables d'état de modéles de guide d'ondes avec dérivation fractionnaire. Paris: Thése IRCAM, 1994.

[14] *Rutman R.* // Theory and practice of geophysical data inversion / ed. Vogel A. et al.: Vieweg, Braunschweig/Wiesbaben, 1992. P. 49–71.

[15] *Andersen R.S.* // J. Inst. Math. Its Appl. 1976. V. 17. P. 329–342; *Bokasten K.J.* // J. Opt. Soc. Amer. 1961. V. 51. P. 943–947; *Maldonaldo C.D., Caron A.P., Olsen H.N.* // J. Opt. Soc. Amer. 1965. V. 55. P. 1247–1254; *Minerbo G.N., Levy M.E.* // SIAM J. Numer. Anal. 1965. V. 6. P. 598–616; *Nestor O.H., Olsen H.N.* // SIAM Rev. 1960. V. 2. P. 200-207.

[16] *Mikhlin S.G.* Multidimensional singular integrals and integral equations. Pergamon Press, 1965.

[17] *Oldham K.B., Spanier J.* The fractional calculus. New York: Academic Press, 1974.

University of Massachusetts Dartmouth
Dartmouth MA 02747, USA

Поступила в редакцию
19.XII.1994 г.

Р.С. Рутман

О ФИЗИЧЕСКИХ ИНТЕРПРЕТАЦИЯХ ФРАКТАЛЬНОГО ИНТЕГРИРОВАНИЯ И ДИФФЕРЕНЦИРОВАНИЯ

Изучена связь между фрактальным исчислением и фрактальной геометрией, а также возможность представления системы дробного порядка причинной динамической моделью. Проанализированы некоторые из недавно предложенных моделей и показано, что они не могут рассматриваться как какие-либо динамические модели систем для дробного порядка, построенных с помощью фракталов. Линейные фильтры с постоянными параметрами имеют очень ограниченное использование в качестве аппроксимаций систем дробного порядка. Модель, предлагаемая в статье, дает представление пространственных состояний с параметрами, являющимися функциями независимых переменных. Рассмотрена регуляризация фрактального дифференцирования и оценена его асимптотическая ошибка. Представлены результаты численных расчетов.