FIBONACCI MODULES AND MULTIPLE FIBONACCI SEQUENCES

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Abstract. Double Fibonacci sequences $(x_{n,k})$ are introduced and they are related to operations with Fibonacci modules. Generalizations and examples are also discussed.

1. Introduction

Let us fix a commutative ring $\mathcal{R}$; $\mathcal{R}^2$ will denote the rank 2 free $\mathcal{R}$-module and also the product ring $\mathcal{R} \times \mathcal{R}$. The main object of study is the Fibonacci module of type $(a, b) \in \mathcal{R}^2$ associated to the $\mathcal{R}$-module $\mathbf{M}$:

Definition 1.1. $\mathcal{F}_{\mathbf{M}}(a, b)$ is the set of sequences $\{(x_n)_{n\geq 0} : x_n \in \mathbf{M},\ x_{n+2} = ax_{n+1}+bx_n, \forall n \geq 0\}$. If $\mathbf{M} = \mathcal{R}$, we use the shorter notation $\mathcal{F}(a, b)$.

Remark 1.2. Using the $\mathcal{R}[T]$ structure of the $\mathcal{R}$-module of all sequences in $\mathbf{M}$: $\mathcal{S}_{\mathbf{M}} = \{(x_n)_{n\geq 0} : x_n \in \mathbf{M}\}$, where the action T is given by the shift $T(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$, one can describe $\mathcal{F}_{\mathbf{M}}(a, b)$ as the sub $\mathcal{R}[T]$-module $\ker(T^2 - aT - b)$. We also consider $\widetilde{\mathcal{F}}_{\mathbf{M}}(a, b) = \{(x_n)_{n\in\mathbb{Z}} : x_n \in \mathbf{M}, x_{n+2} = ax_{n+1} + bx_n, \forall n\}$.

It is well known (at least in the vector space case) that $\mathcal{F}(a, b)$ is a free $\mathcal{R}$-module of rank 2; more generally:

Proposition 1.3.
$$\mathcal{F}_{\mathbf{M}}(a, b) \cong \mathbf{M} \oplus \mathbf{M} \cong \mathcal{F}(a, b) \otimes \mathbf{M}.$$

An explicit basis can be found for $\mathcal{F}_{\mathbf{M}}(a, b)$ (see, for example, [2] in which Lucas functions are used):

Proposition 1.4. *The sequences* $\left(P_0^{[n]}(a, b)\right)_{n\geq 0}$ *and* $\left(P_1^{[n]}(a, b)\right)_{n\geq 0}$ *in* $\mathcal{F}(a, b)$ *defined by* $P_0^{[0]}(a, b) = 1$, $P_0^{[1]}(a, b) = 0$, *respectively by* $P_1^{[0]}(a, b) = 0$, $P_1^{[1]}(a, b) = 1$, *and by* $P_i^{[n+2]}(a, b) = aP_i^{[n+1]}(a, b) + bP_i^{[n]}(a, b)$ $(i = 0, 1)$ *give a canonical basis of the* $\mathcal{R}$*-module* $\mathcal{F}(a, b)$.

Standard operations with modules give the following:

Proposition 1.5. ***a***) *There is a natural* $\mathcal{R}[T]$*-module isomorphism:*
$$\mathcal{F}_{\mathbf{M}}(a, b) \oplus \mathcal{F}_{\mathbf{N}}(a, b) \cong \mathcal{F}_{\mathbf{M}\oplus\mathbf{N}}(a, b)\,.$$
b) *There is a natural* $\mathcal{R}^2$*-module isomorphism:*
$$\mathcal{F}_{\mathbf{M}}(a, b) \oplus \mathcal{F}_{\mathbf{N}}(c, d) \cong \mathcal{F}_{\mathbf{M}\oplus\mathbf{N}}\big((a, c), (b, d)\big)\,.$$

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In order to describe multiplicative operations (tensor product, symmetric power, exterior power), we introduce double Fibonacci sequences.

Definition 1.6. The double sequence $(x_{n,k})_{n,k\geq 0}$, $x_{n,k} \in \mathbf{M}$ is a double Fibonacci sequence of type $(a,b)\otimes(c,d) \in \mathcal{R}^2 \otimes \mathcal{R}^2$ if for any $n,k \geq 0$ we have:

$$x_{n+2,k} = ax_{n+1,k} + bx_{n,k}\ ,$$
$$x_{n,k+2} = cx_{n,k+1} + dx_{n,k}\ .$$

As an example, let us consider the element in $\mathcal{F}^{[2]}_{\mathbb{Z}}\big((1,1)\otimes(1,3)\big)$ with $x_{0,0} = x_{1,0} = x_{1,1} = 1$ and $x_{0,1} = 0$ (we locate the terms in the first quadrant):

$$\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \\ 3 & 7 & 10 & 17 & \cdots \\ 3 & 4 & 7 & 11 & \cdots \\ 0 & 1 & 1 & 2 & \cdots \\ 1 & 1 & 2 & 3 & \cdots \end{array}$$

The set of double Fibonacci sequences is denoted by $\mathcal{F}^{[2]}_{\mathbf{M}}\big((a,b)\otimes(c,d)\big)$ and it is naturally an $\mathcal{R}[H,V]$-module $\big(H,\ V$ are horizontal and vertical shifts: $H(x_{n,k}) = (x_{n+1,k})$, respectively $V(x_{n,k}) = (x_{n,k+1})\big)$. If $(a,b) = (c,d)$ we use the simplified notation $\mathcal{F}^{[2]}_{\mathbf{M}}(a,b)$. In [3] double sequences $(x_{n,k})$ given by a different recurrency are considered: $x_{n,k}$ depends linearly on the terms $\{x_{i,j}\}_{i+j<n+k}$. In our definition, $x_{n,k}$ depends on $x_{n-1,k}$ and $x_{n-2,k}$ and also depends on $x_{n,k-1}$ and $x_{n,k-2}$, using two different relations. Even the existence of a sequence with prescribed initial four terms $x_{i,j}$, $(i,j) \in \{0,1\}^2$, is not an obvious fact. Now we present some properties and operations with these sequences.

In Section 2 the proofs of the previous results are given. In Section 3 we generalize these results in two directions: we consider higher order linear recurrency:

$$x_{n+d} = a_1 x_{n+d-1} + \cdots + a_d x_n,$$

and also we consider multiple sequences: $(x_{n_1,n_2,\ldots,n_d})_{n_i\geq 0}$.

In the last section examples of double Fibonacci sequences are given and also an interesting property of their diagonals is presented.

Proposition 1.7. *There is a natural isomorphism of* $\mathcal{R}[H,V]$*-modules:*

$$\mathcal{F}_{\mathbf{M}}(a,b)\otimes_{\mathcal{R}}\mathcal{F}_{\mathbf{N}}(c,d) \cong \mathcal{F}^{[2]}_{\mathbf{M}\otimes\mathbf{N}}\big((a,b)\otimes(c,d)\big)\ .$$

Corollary 1.8. *The module* $\mathcal{F}^{[2]}\big((a,b)\otimes(c,d)\big)$ *is a free* $\mathcal{R}$*-module of rank* 4*. In general,* $\mathcal{F}^{[2]}_{\mathbf{M}\otimes\mathbf{N}}\big((a,b)\otimes(c,d)\big)$ *is isomorphic to* $(\mathbf{M}\otimes\mathbf{N})^4$.

An explicit basis of $\mathcal{F}^{[2]}\big((a,b)\otimes(c,d)\big)$ is given by the four sequences $\big(P^{[n,k]}_{i,j}(a,b)\otimes(c,d)\big)_{n,k\geq 0} = \big(P^{[n]}_i(a,b)P^{[k]}_j(c,d)\big)_{n,k\geq 0}$, where $(i,j) \in \{0,1\}^2$.

The generating function of a double sequence $(x_{n,k})_{n,k} \geq 0$ is the formal series in $\mathcal{R}[[t,s]]\otimes\mathbf{M} \cong \mathbf{M}[[t,s]]$:

$$G(t,s) = x_{0,0} + x_{1,0}t + x_{0,1}s + \cdots + x_{n,k}t^n s^k + \cdots\ .$$

Proposition 1.9. *A Fibonacci sequence* $(x_{n,k})$ *of type* $(a,b)\otimes(c,d)$ *has a rational generating function given by*

$$G(t,s) = q(t)^{-1}r(s)^{-1}\left[x_{0,0}(1-at)(1-cs) + x_{1,0}t(1-cs) + x_{0,1}(1-at)s + x_{1,1}ts\right]$$

where $q(t) = 1 - at - bt^2$, $r(s) = 1 - cs - ds^2$.

2. Proofs

We can write well-known results on Fibonacci sequences in the following form:

Lemma 2.1. *There are polynomials* $P_0^{[n]}, P_1^{[n]} \in \mathcal{R}[T,U]$ *such that for any* $(x_n)_{n\geq 0} \in \mathcal{F}_{\mathbf{M}}(a,b)$:

$$x_n = P_0^{[n]}(a,b)x_0 + P_1^{[n]}(a,b)x_1 \tag{2.1}$$

for every $n \geq 0$.

Proof. We define $P_0^{[0]} = 1$, $P_0^{[1]} = 0$ and $P_1^{[0]} = 0$, $P_1^{[1]} = 1$, and $P_i^{[n+2]} = aP_i^{[n+1]} + bP_i^{[n]}$ $(i = 0,1)$. These satisfy the equation (2.1) by definition for $n = 0,1$ and by induction for $n \geq 2$. □

Remark 2.2. The Lemma 2.1 shows that the $\mathcal{R}$-module $\mathcal{F}(a,b)$ is free of rank 2 with basis $\big(P_0^{[n]}(a,b)\big)_{n\geq 0}$, $\big(P_1^{[n]}(a,b)\big)_{n\geq 0}$.

Remark 2.3. If $a = r_1 + r_2$, $b = -r_1r_2$ then one can describe $P_0^{[n]}$ and $P_1^{[n]}$ in the classical way as polynomials in r_1, r_2:

$$\begin{aligned} P_0^{[n]}(r_1+r_2, -r_1r_2) &= R_0^{[n]}(r_1,r_2) = -r_1^{n-1}r_2 - r_1^{n-2}r_2^2 - \cdots - r_1r_2^{n-1}, \\ P_1^{[n]}(r_1+r_2, -r_1r_2) &= R_1^{[n]}(r_1,r_2) = r_1^{n-1} + r_1^{n-2}r_2^1 + \cdots + r_2^{n-1}, \end{aligned}$$

or as rational functions in r_1, r_2:

$$R_0^{[n]}(r_1,r_2) = \frac{r_1^n r_2 - r_1 r_2^n}{r_2 - r_1}\,, \quad R_1^{[n]}(r_1,r_2) = \frac{r_2^n - r_1^n}{r_2 - r_1}\,. \tag{2.2}$$

Remark 2.4. The previous formulae are also correct in $\widetilde{\mathcal{F}}_{\mathbf{M}}(a,b)$, *i.e.* for negative n, if we extend the scalars to a suitable ring of fractions.

For an arbitrary sequence $(x_n)_{n\geq 0}$ in $\mathcal{S}_{\mathbf{M}}$ we define its *generating function* $G(t)$ as a formal series in $\mathcal{R}[[t]] \otimes \mathbf{M} \cong \mathbf{M}[[t]]$:

$$G(t) = x_0 + x_1 t + x_2 t^2 + \cdots\,.$$

Another classical result is (see, for example, [1]):

Lemma 2.5. *The generating function of the Fibonacci sequence* $(x_n)_{n\geq 0} \in \mathcal{F}_{\mathbf{M}}(a,b)$ *is the rational function*

$$G(t) = \frac{(1-at)x_0 + tx_1}{1 - at - bt^2} = q(t)^{-1}\big[x_0 + (x_1 - ax_0)t\big]\,,$$

where $q(t) = 1 - at - bt^2$.

Proof. [Proposition 1.4] From Lemma 2.1, an arbitrary sequence $(x_n)_{n\geq 0} \in \mathcal{F}(a,b)$ can be written as $(x_n)_{n\geq 0} = \big(P_0^{[n]}(a,b)\big)_{n\geq 0}\, x_0 + \big(P_1^{[n]}(a,b)\big)_{n\geq 0}\, x_1$. □

Proof. [Proposition 1.3] Define the morphisms

$$\mathcal{F}_{\mathbf{M}}(a,b) \xrightarrow{\varphi} \mathbf{M} \oplus \mathbf{M} \xrightarrow{\psi} \mathcal{F}(a,b) \otimes \mathbf{M} \xrightarrow{\eta} \mathcal{F}_{\mathbf{M}}(a,b)$$

by

$$\begin{gathered} \varphi\big((x_n)_{n\geq 0}\big) = (x_0, x_1), \\ \psi(x_0,x_1) = \big(P_0^{[n]}(a,b)\big)_{n\geq 0} \otimes x_0 + \big(P_1^{[n]}(a,b)\big)_{n\geq 0} \otimes x_1, \end{gathered}$$

and

$$\eta\big((c_n)_{n\geq 0}\otimes x\big) = (c_n x)_{n\geq 0}.$$

It is easy to check that $\eta\psi\varphi$, $\varphi\eta\psi$ and $\psi\varphi\eta$ are identities, so φ, ψ, η are $\mathcal{R}$-module isomorphisms. It is also obvious that η and $\psi\varphi$ are $\mathcal{R}[T]$-linear. □

Proof. [Proposition 1.5] There are canonical maps:

$$\Phi : \mathcal{F}_{\mathbf{M}}(a,b)\oplus\mathcal{F}_{\mathbf{N}}(a,b)\longrightarrow\mathcal{F}_{\mathbf{M}\oplus\mathbf{N}}(a,b)$$

defined by

$$\Phi\big((x_n)_{n\geq 0},(y_n)_{n\geq 0}\big) = (x_n,y_n)_{n\geq 0}$$

and

$$\Psi : \mathcal{F}_{\mathbf{M}}(a,b)\oplus\mathcal{F}_{\mathbf{N}}(c,d)\longrightarrow\mathcal{F}_{\mathbf{M}\oplus\mathbf{N}}\big((a,c),(b,d)\big)$$

defined by

$$\Psi\big((x_n)_{n\geq 0},(y_n)_{n\geq 0}\big) = (x_n,y_n)_{n\geq 0}\,.$$

Both are compatible with the shift. □

Proof. [Proposition 1.7] Define the morphism of $\mathcal{R}[H,V]$-modules:

$$\Phi : \mathcal{F}_{\mathbf{M}}(a,b)\otimes\mathcal{F}_{\mathbf{N}}(c,d)\longrightarrow\mathcal{F}^{[2]}_{\mathbf{M}\otimes\mathbf{N}}\big((a,b)\otimes(c,d)\big)$$

by

$$\Phi\big((x_n)_{n\geq 0}\otimes(y_k)_{k\geq 0}\big) = (x_n\otimes y_k)_{n,k\geq 0}\,.$$

The inverse morphism Ψ can be constructed using canonical bases $P_0^{[n]}(a,b)$, $P_1^{[n]}(a,b)$ of $\mathcal{F}_{\mathbf{M}}(a,b)$, respectively $P_0^{[k]}(c,d)$, $P_1^{[k]}(c,d)$ of $\mathcal{F}_{\mathbf{N}}(c,d)$ and the corresponding basis $P_i^{[n]}(a,b)\otimes P_j^{[k]}(c,d)$, $i,j\in\{0,1\}^2$ of $\mathcal{F}_{\mathbf{M}}(a,b)\otimes\mathcal{F}_{\mathbf{N}}(c,d)$: if the first four terms are given by $Z_{0,0}=\sum_{i\in I} m_i\otimes n_i$, $Z_{1,0}=\sum_{j\in J} m_j^{'}\otimes n_j^{'}$, $Z_{0,1}=\sum_{h\in H} m_h^{''}\otimes n_h^{''}$, $Z_{1,1}=\sum_{l\in L} m_l^{'''}\otimes n_l^{'''}$, then Ψ is defined by:

$$\begin{aligned}\Psi\big((Z_{n,k})_{n,k\geq 0}\big) = \;& \textstyle\sum_{i\in I}\big(P_0^{[n]}(a,b)m_i\big)_{n\geq 0}\otimes\big(P_0^{[k]}(c,d)n_i\big)_{k\geq 0}\\ &+\textstyle\sum_{j\in J}\big(P_1^{[n]}(a,b)m_j^{'}\big)_{n\geq 0}\otimes\big(P_0^{[k]}(c,d)n_j^{'}\big)_{k\geq 0}\\ &+\textstyle\sum_{h\in H}\big(P_0^{[n]}(a,b)m_h^{''}\big)_{n\geq 0}\otimes\big(P_1^{[k]}(c,d)n_h^{''}\big)_{k\geq 0}\\ &+\textstyle\sum_{l\in L}\big(P_1^{[n]}(a,b)m_l^{'''}\big)_{n\geq 0}\otimes\big(P_1^{[k]}(c,d)n_l^{'''}\big)_{k\geq 0}\,.\end{aligned}$$

□

Proof. [Corollary 1.8] The proof is clear as $\mathcal{F}^{[2]}\big((a,b)\otimes(c,d)\big)\cong\mathcal{F}(a,b)\otimes\mathcal{F}(c,d)\cong(\mathcal{R}\oplus\mathcal{R})\otimes(\mathcal{R}\oplus\mathcal{R})\cong\mathcal{R}^4$. In general, $\mathcal{F}^{[2]}_{\mathbf{M}\otimes\mathbf{N}}\big((a,b)\otimes(c,d)\big)\cong\mathcal{F}_{\mathbf{M}}(a,b)\otimes\mathcal{F}_{\mathbf{N}}(c,d)\cong(\mathbf{M}\oplus\mathbf{M})\otimes(\mathbf{N}\oplus\mathbf{N})\cong(\mathbf{M}\otimes\mathbf{N})^4$. □

Corollary 2.6. *Using $a=r_1+r_2$, $b=-r_1r_2$, the general term $x_{n,k}$ of a sequence in $\mathcal{F}^{[2]}_{\mathbf{M}\otimes\mathbf{N}}(a,b)$ is given by*

$$\begin{aligned}x_{n,k} = \Delta^{-2}\big[&(r_1^n r_2 - r_1 r_2^n)(r_1^k r_2 - r_1 r_2^k)x_{0,0} + (r_2^n - r_1^n)(r_1^k r_2 - r_1 r_2^k)x_{1,0}\\ &+ (r_1^n r_2 - r_1 r_2^n)(r_2^k - r_1^k)x_{0,1} + (r_2^n - r_1^n)(r_2^k - r_1^k)x_{1,1}\big],\end{aligned}$$

where $\Delta = r_2 - r_1$. This formula is correct for arbitrary integers n,k $\big($as an equality in the ring $\mathcal{R}(r_1,r_2)$ of rational functions$\big)$.

Proof. [Proposition 1.9] Apply two times Lemma 2.5:

$$\begin{aligned} G(t,s) = & \sum_{n\geq 0}\big(\sum_{k\geq 0} x_{n,k}s^k\big)t^n \\ = & \sum_{n\geq 0}\big[r(s)^{-1}x_{n,0}(1-cs)+r(s)^{-1}x_{n,1}s\big]t^n \\ = & \; r(s)^{-1}\big[(1-cs)\sum_{n\geq 0} x_{n,0}t^n + s\sum_{n\geq 0} x_{n,1}t^n\big] \\ = & \; q(t)^{-1}r(s)^{-1}\big\{(1-cs)[x_{0,0}(1-at)+x_{1,0}t] \\ & +s[x_{0,1}(1-at)+x_{1,1}t]\big\}\,. \end{aligned}$$

□

We consider also other operations with Fibonacci modules, for example symmetric powers and exterior products (we suppose that 2 is a unit in $\mathcal{R}$):

Proposition 2.7. *There are natural isomorphisms:*

$$\mathrm{Symm}^{(2)}\,\mathcal{F}_{\mathbf{M}}(a,b) \cong \big\{(x_{n,k}) \in \mathcal{F}^{[2]}_{\mathbf{M}\otimes\mathbf{N}}(a,b):\ x_{n,k}=x_{k,n}\ \ \forall\ k,n\geq 0\big\},$$

$$\wedge^{(2)}\,\mathcal{F}_{\mathbf{M}}(a,b) \cong \big\{(x_{n,k}) \in \mathcal{F}^{[2]}_{\mathbf{M}\otimes\mathbf{N}}(a,b):\ x_{n,k}=-x_{k,n}\ \ \forall\ k,n\geq 0\big\}.$$

3. Generalizations

First we introduce recurrency of order d:

Definition 3.1. Let $\mathbf{a}=(a_1,\dots,a_d)$ be an element in $\mathcal{R}^d$. The Fibonacci module of type $\mathbf{a}$ associated to the module $\mathbf{M}$ is the $\mathcal{R}[T]$-module:

$$\mathcal{F}_{\mathbf{M}}(\mathbf{a}) = \big\{(x_n)_{n\geq 0} \in \mathcal{S}_{\mathbf{M}}:\ x_{n+d} = a_1x_{n+d-1}+\cdots+a_dx_n,\ \forall\ n\geq 0\big\}.$$

Next we consider multiple Fibonacci sequences $(x_{n_1,\dots,n_p})_{n_i\geq 0}$ in $\mathbf{M}$:

Definition 3.2. Let $\mathbf{a}^{(1)}\in\mathcal{R}^{d_1},\dots,\mathbf{a}^{(p)}\in\mathcal{R}^{d_p}$. The Fibonacci module of type $(\mathbf{a}^{(1)},\dots,\mathbf{a}^{(p)})$ associated to the module $\mathbf{M}$ is the $\mathcal{R}[T_1,\dots,T_p]$-module:

$$\begin{aligned}\mathcal{F}^{[p]}_{\mathbf{M}}(\mathbf{a}^{(1)},\dots,\mathbf{a}^{(p)}) = & \big\{(x_{n_1,\dots,n_p})_{n_i\geq 0} : x_{n_1,\dots,n_p}\in\mathbf{M},\ x_{n_1,\dots,n_i+d_i,\dots,n_p} = \\ & \textstyle\sum_{j=1}^{d_i} a^{(i)}_j x_{n_1,\dots,n_i+d_i-j,\dots,n_p} \text{ for } i=1,2,\dots,p\big\}.\end{aligned}$$

If $\mathbf{a}^{(1)}=\cdots=\mathbf{a}^{(p)}=\mathbf{a}=(a_1,\dots,a_d)$, we denote simply $\mathcal{F}^{[p]}_{\mathbf{M}}(\mathbf{a})=\mathcal{F}^{[p]}_{\mathbf{M}}(a_1,\dots,a_d)\,.$

The previous results have obvious generalizations. For example:

Proposition 3.3.

$$\mathcal{F}_{\mathbf{M}}(a_1,\dots,a_d)\cong\mathbf{M}^d\cong\mathcal{F}(a_1,\dots,a_d)\otimes\mathbf{M}.$$

Proposition 3.4. *Fix* $\mathbf{a}=(a_1,\dots,a_d)\in\mathcal{R}^d$. *The sequences* $\big(P^{[n]}_i(\mathbf{a})\big)_{n\geq 0}$, $i=0,\dots,d-1$ *in* $\mathcal{F}(\mathbf{a})$ *defined by* $P^{[j]}_i(\mathbf{a})=\delta_{ij}$ *(for* $j=0,\dots,d-1$*) give a canonical basis of* $\mathcal{F}(\mathbf{a})$.

Lemma 3.5. *The generating function of* $(x_n)_{n\geq 0}$ *in* $\mathcal{F}(\mathbf{a})$ *is*

$$G(t)=q(t)^{-1}\big[Q_0(t)x_0+Q_1(t)x_1+\cdots+Q_{d-1}(t)x_d\big],$$

where

$$Q_i(t)=t^i\big(1-a_1t-a_2t^2-\cdots-a_{d-i-1}t^{d-i-1}\big),\ i\in\{0,\dots,d-1\},$$

and $q(t)=1-a_1t-a_2t^2-\cdots-a_dt^d\,.$

Proposition 3.6.

$$\mathcal{F}_{\mathbf{M_1}}(\mathbf{a}^{(1)}) \otimes \cdots \otimes \mathcal{F}_{\mathbf{M_p}}(\mathbf{a}^{(p)}) \cong \mathcal{F}^{[p]}_{\mathbf{M_1}\otimes\cdots\otimes\mathbf{M_p}}(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(p)})\,.$$

In particular, $\mathcal{F}^{[p]}(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(p)})$ *is free of rank* $D = d_1 d_2 \cdots d_p$.

Proposition 3.7. *A multiple Fibonacci sequence* $(x_{n_1,\ldots,n_p})$ *of type* $(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(p)})$ *has a rational generating function:*

$$G(t_1, \ldots, t_p) = q_1(t_1)^{-1} \cdots q_p(t_p)^{-1} \Big[\sum_{0 \le j_i \le d_i - 1} Q^{(1)}_{j_1}(t_1) \cdots Q^{(p)}_{j_p}(t_p) x_{j_1,\ldots,j_p} \Big],$$

where $q_i(t) = 1 - a^{(i)}_1 t - \cdots - a^{(i)}_{d_i} t^{d_i}$ *and* $Q^{(i)}_0, \ldots, Q^{(i)}_{d_i-1}$ *are like in Lemma 3.5.*

For further applications in knot theory, we will use the next specializations:

Theorem 3.8. *Let* $(x_{n_1,\ldots,n_p})_{\ge 0}$ *be an element in* $\mathcal{F}^{[p]}_{\mathbf{M}}(r_1 + r_2, -r_1 r_2)$.
a) *The general term is given by*

$$x_{n_1,\ldots,n_p} = \Delta^{-p} \sum_{0 \le j_1,\ldots,j_p \le 1} S^{[n_1]}_{j_1}(r_1, r_2) \cdots S^{[n_p]}_{j_p}(r_1, r_2) x_{j_1,\ldots,j_p}\,,$$

where $\Delta = r_2 - r_1$, $S^{[n]}_0(r_1, r_2) = r_1^n r_2 - r_1 r_2^n$, $S^{[n]}_1(r_1, r_2) = r_2^n - r_1^n$;
b) *the generating function of* $(x_{n_1,\ldots,n_p})$ *is given by*

$$G(t_1, \ldots, t_p) = q(t_1)^{-1} \cdots q(t_p)^{-1} \sum_{0 \le j_1,\ldots,j_p \le 1} Q_{j_1}(t_1) \cdots Q_{j_p}(t_p) x_{j_1,\ldots,j_p}\,,$$

where $q(t) = (1 - r_1 t)(1 - r_2 t)$, $Q_0(t) = 1 - (r_1 + r_2)t$ *and* $Q_1(t) = t$.

4. **Examples**

Example 4.1. Fibonacci module $\mathcal{F}^{[2]}_{\mathbb{Z}}(1, 1)$: let us analyze sequences with the first four entries $(c_{i,j})_{(i,j)\in\{0,1\}^2}$ equal to 0 or 1. From the sixteen possible choices there are 5 primitive sequences:

$$B_0 = \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}},\ B_1 = \boxed{\begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}},\ B_2 = \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}},\ B_3 = \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}},\ B_4 = \boxed{\begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}}.$$

The others are shifts of these primitive sequences (see figure below):

$$H(B_1) = \boxed{\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}},\ H^2(B_1) = \boxed{\begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix}},\ V(B_1) = \boxed{\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}},\ V^2(B_1) = \boxed{\begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix}},$$
$$HV(B_1) = \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}}, H^2V(B_1) = \boxed{\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}}, HV^2(B_1) = \boxed{\begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix}}, H^2V^2(B_1) = \boxed{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}}.$$

$$H(B_2) = \boxed{\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}}, V(B_2) = \boxed{\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}} \text{ and } H(B_3) = V(B_3) = \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}}.$$

In fact, using the structure of $\mathbb{Z}[H, V]$-module, $\mathcal{F}^{[2]}_{\mathbb{Z}}(1, 1)$ is generated by B_1.

It is obvious that an element $(x_n)_{n\ge 0} \in \mathcal{F}_{\mathbb{Q}}(1, 1)$ can be defined by any two terms $\{x_p, x_q\}$; in the case of a double sequence $(x_{n,k})_{n,k\ge 0} \in \mathcal{F}^{[2]}_{\mathbb{Q}}(1, 1)$, not any four terms $\{x_{l,m}, x_{p,q}, x_{r,s}, x_{u,v}\}$ can define the sequence.

13								
8	0							
5	0	5						
3	0	3	3					
2	0	2	2	4				
1	0	1	1	2	3			
1	0	1	1	2	3	5		
0	0	0	0	0	0	0	0	
1	0	1	1	2	3	5	8	13

21								
13	8							
8	5	13						
5	3	8	11					
3	2	5	7	12				
2	1	3	4	7	11			
1	1	2	3	5	8	13		
1	0	1	1	2	3	5	8	
0	1	1	2	3	5	8	13	21

A curious property of these sequences is the alternating monotonicity along the lines parallel to the secondary diagonal:

$$x_{n+2,k} \geq x_{n+1,k+1} \leq x_{n,k+2}$$

or

$$x_{n+2,k} \leq x_{n+1,k+1} \geq x_{n,k+2}\,.$$

In general we do not have this strong alternating property (look at the sequence given by $x_{0,0} = x_{1,0} = 3, x_{0,1} = 2, x_{1,1} = 0$: the 4th diagonal is $(7,3,2,9)$). In general we have only a "weak alternating property":

$$x_{n+2,k+1} \geq x_{n+1,k+2} \text{ if and only if } x_{n+3,k} \leq x_{n,k+3}$$

(see the next corollary).

The general statement explaining these two facts is given by:

Proposition 4.2. (diagonal property) *If $a^2d = bc^2$, any four diagonal consecutive terms of the sequence $(x_{n,k})_{n,k\geq 0} \in \mathcal{F}^{[2]}_{\mathbf{M}}\big((a,b)\otimes(c,d)\big)$ satisfy the relation:*

$$abx_{n,k+3} + (a^2+b)cx_{n+1,k+2} = a(c^2+d)x_{n+2,k+1} + cdx_{n+3,k}\,.$$

Proof. Express the terms as combinations of $x_{n,k}$, $x_{n+1,k}$, $x_{n,k+1}$ and $x_{n+1,k+1}$. □

Corollary 4.3. *Four diagonal consecutive terms in $(x_{n,k})_{n,k\geq 0} \in \mathcal{F}^{[2]}_{\mathbb{Z}}(1,1)$ satisfy*

$$x_{n,k+3} - x_{n+3,k} = 2(x_{n+2,k+1} - x_{n+1,k+2})\,.$$

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