THE FRACTIONAL POISSON PROCESS AND THE INVERSE STABLE SUBORDINATOR

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Abstract. The fractional Poisson process is a renewal process with Mittag-Leffler waiting times. Its distributions solve a time-fractional analogue of the Kolmogorov forward equation for a Poisson process. This paper shows that a traditional Poisson process, with the time variable replaced by an independent inverse stable subordinator, is also a fractional Poisson process. This result unifies the two main approaches in the stochastic theory of time-fractional diffusion equations. The equivalence extends to a broad class of renewal processes that include models for tempered fractional diffusion, and distributed-order (e.g., ultraslow) fractional diffusion.

1. Introduction

The fractional Poisson process (FPP) is a natural generalization of the usual Poisson process, with an interesting connection to fractional calculus. Mainardi et al. [24] define the FPP as a renewal process whose IID waiting times J_n satisfy

$$\mathbb{P}(J_n > t) = E_\beta(-\lambda t^\beta) \tag{1.1}$$

for $0 < \beta \leq 1$, where

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}. \tag{1.2}$$

denotes the Mittag-Leffler function. When $\beta = 1$, the waiting times are exponential with rate λ, since $e^z = E_1(z)$. Let $T_n = J_1 + \cdots + J_n$ be the time of the nth jump. The FPP

$$N_\beta(t) = \max\{n \geq 0 : T_n \leq t\} \tag{1.3}$$

is a renewal process with Mittag-Leffler waiting times. A compound FPP is obtained by subordinating a random walk to the FPP. The resulting process is non-Markovian (unless $\beta = 1$) and the distribution of that process solves a "master equation" analogous to the Kolmogorov equation for Markov processes, with the usual integer order time derivative replaced by a fractional derivative.

The continuous time random walk (CTRW) is another useful model in fractional calculus. Consider a CTRW whose IID particle jumps Y_n have PDF $w(x)$, and whose IID waiting times (J_n) are Mittag-Leffler variables independent of (Y_n). The particle

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location after n jumps is $S(n) = Y_1 + \cdots + Y_n$, and the CTRW $S(N_\beta(t))$ gives the particle location at time $t \geq 0$. Hilfer and Anton [16] show that the PDF $p(x,t)$ of the CTRW $S(N_\beta(t))$ solves the fractional master equation

$$\partial_t^\beta p(x,t) = -\lambda p(x,t) + \lambda \int_{-\infty}^{\infty} p(x-y,t) w(y)\, dy \tag{1.4}$$

where ∂_t^β denotes the Caputo fractional derivative. The Caputo fractional derivative, defined for $0 \leq n-1 < \beta < n$ by

$$\partial_t^\beta g(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-r)^{n-1-\beta} g^{(n)}(r) dr, \tag{1.5}$$

where $g^{(k)}$ denotes the k-th derivative of g, was invented to properly handle initial values [8].

If $\beta = 1$, then ∂_t^β is the usual first derivative. The corresponding CTRW $S(N_1(t))$ is a compound Poisson process, and (1.4) reduces to

$$\partial_t p(x,t) = -\lambda p(x,t) + \lambda \int_{-\infty}^{\infty} p(x-y,t) w(y)\, dy, \tag{1.6}$$

the Cauchy problem associated with this infinitely divisible Lévy process. Then a general result on Cauchy problems [2, Theorem 3.1] implies that the PDF of the time-changed process $S(N_1(E(t)))$ solves the fractional Cauchy problem (1.4), where

$$E(t) = \inf\{r > 0 : D(r) > t\} \tag{1.7}$$

is the right-continuous inverse (hitting time, first passage time) of $D(t)$, a standard β-stable subordinator with $\mathbb{E}[e^{-sD(t)}] = e^{-ts^\beta}$ for some $0 < \beta < 1$.

Since the PDF of both $S(N_\beta(t))$ and $S(N_1(E(t)))$ solve the same governing equation (1.4), with the same point-source initial condition (i.e., both processes start at the origin), these two processes have the same one dimensional distributions. Heuristically, the degenerate case $Y_n \equiv 1$ gives $S(n) = n$, which strongly suggest that the FPP $N_\beta(t)$ and the process $N_1(E(t))$ have the same one dimensional distributions. We will call $N_1(E(t))$ the fractal time Poisson process (FTPP), since it comes from a self-similar time change (see, e.g., [26, Proposition 3.1]). In this paper, we will prove that the FPP and the FTPP are in fact the same process, by showing that the waiting times between jumps in the FTPP are IID Mittag-Leffler. This strong connection between the FPP and the FTPP unifies the two main approaches in the stochastic theory of fractional diffusion. For example, the FPP approach was used recently in the work of Behgin and Orsingher [5], while the inverse stable subordinator is a key ingredient in [30].

2. Two equivalent formulations

Recall that the fractional Poisson process (FPP) $N_\beta(t)$ is a renewal process with Mittag-Leffler waiting times (1.1), and the fractal time Poisson process (FTPP)

$N_1(E(t))$ is Poisson process, with rate $\lambda > 0$, time-changed via the inverse stable subordinator (1.7). The proof that the FPP and the FTPP are the same process requires the following simple lemma.

Lemma 2.1. *Let $D(t)$ be a strictly increasing right-continuous process with left-hand limits, and let $E(t)$ be its right-continuous inverse defined by* (1.7). *Then*

$$D(r-) = \sup\{t > 0 : E(t) < r\} \tag{2.1}$$

for any $r > 0$.

Proof. Let $t_0 = \sup\{t > 0 : E(t) < r\}$. Then there exists a sequence of points $t_n \uparrow t_0$ such that $E(t_n) < r$ for all n. Let $\varepsilon_n = r - E(t_n) > 0$. If $r > E(t)$ then, since $D(t)$ is strictly increasing, $D(r) > t$. Since $D(r)$ is right-continuous, it follows that $D(E(t)) \geq t$ for all $t > 0$. Then we have $t_n \leq D(E(t_n)) = D(r - \varepsilon_n) < D(r-)$. Letting $n \to \infty$ shows that $D(r-) \geq t_0$.

Since D has left-hand limits, for any $r_n \uparrow r$ we have $D(r_n) \to D(r-)$ as $n \to \infty$. If $D(r-) > t_0$, then for some $r_n < r$ we have $D(r_n) > t_0$. Since $E(t)$ is nondecreasing and continuous, this implies that $E(D(r_n)) \geq r$, by definition of t_0. But, $E(D(r)) = r$ for all $r > 0$ implying that $r_n \geq r$, which is a contradiction. Thus, (2.1) follows. □

Theorem 2.2. *For any $0 < \beta < 1$, the FTPP $N_1(E(t))$ is also a FPP. That is, the waiting times between jumps of the FTPP are IID Mittag-Leffler.*

Proof. Let W_n be an IID sequence with $\mathbb{P}(W_n > t) = e^{-\lambda t}$ and $V_n = W_1 + \cdots + W_n$ so that the Poisson process $N_1(t) = \max\{n \geq 0 : V_n \leq t\}$. Let

$$\tau_n = \sup\{t > 0 : N_1(E(t)) < n\} \tag{2.2}$$

denote the jump times of the FTPP. This definition of the jump times takes into account the fact that $E(t)$ has constant intervals corresponding to the jumps of the process $D(t)$. Using the fact that $\{N_1(t) < n\} = \{V_n > t\}$ for the Poisson process, along with (2.2), we have

$$\tau_n = \sup\{t > 0 : E(t) < V_n\}.$$

Then Lemma 2.1 implies that $\tau_n = D(V_n-)$. Define $X_1 = \tau_1$ and $X_n = \tau_n - \tau_{n-1}$ for $n \geq 2$, the waiting times between jumps of the FTPP. In order to show that the FTPP is an FPP, it suffices to show that X_n are IID Mittag-Leffler, i.e., they are IID with J_n.

Recall that the Laplace transform of the exponential distribution $\mathbb{E}(e^{-sW_n}) = \lambda/(\lambda + s)$. Also recall that $\mathbb{E}(e^{-sD(t)}) = e^{-ts^\beta}$. Since $D(t)$ is a Lévy process, it has no fixed points of discontinuity and hence $D(t-), D(t)$ are identically distributed for all $t \geq 0$. (Indeed, $D(t) = D(t-)$ *a.s.* [1, Lemma 2.3.2]).

Then a conditioning argument yields

$$\begin{aligned} \mathbb{E}(e^{-s\tau_1}) &= \mathbb{E}(e^{-sD(W_1-)}) = \mathbb{E}\left[\mathbb{E}\left(e^{-sD(W_1-)} \middle| W_1\right)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left(e^{-sD(W_1)} \middle| W_1\right)\right] = \mathbb{E}\left[e^{-W_1 s^\beta}\right] = \frac{\lambda}{\lambda + s^\beta}. \end{aligned} \tag{2.3}$$

Let $f_\beta(x) = \partial_x[1 - E_\beta(-\lambda x^\beta)]$ be the Mittag-Leffler PDF of J_n. It is well known that

$$\int_0^\infty e^{-sx} E_\beta(-\lambda x^\beta)\, dx = \frac{s^{\beta-1}}{\lambda + s^\beta},$$

see for example [30, Eq. (3.4)]. Now integrate by parts to see that

$$\begin{aligned}
\mathbb{E}(e^{-sT_1}) &= \int_0^\infty e^{-sx} f_\beta(x)\, dx \\
&= \int_0^\infty s e^{-sx} \left(1 - E_\beta(-\lambda x^\beta)\right)\, dx \qquad (2.4)\\
&= s\left[\frac{1}{s} - \frac{s^{\beta-1}}{\lambda + s^\beta}\right] = \frac{\lambda}{\lambda + s^\beta} = \mathbb{E}(e^{-s\tau_1})
\end{aligned}$$

and then the uniqueness theorem for LT implies that T_1, τ_1 are identically distributed. In particular, X_1 has the same Mittag-Leffler distribution as J_1.

A straightforward extension of this argument shows that $(T_1, \ldots, T_n)$ is identically distributed with $(\tau_1, \ldots, \tau_n)$ for any positive integer n. To ease notation, we only write the case $n = 2$. First observe that

$$\mathbb{E}(e^{-s_1 T_1} e^{-s_2 T_2}) = \mathbb{E}(e^{-s_1 J_1} e^{-s_2(J_1+J_2)}) = \frac{\lambda}{\lambda + (s_1+s_2)^\beta} \cdot \frac{\lambda}{\lambda + s_2^\beta},$$

using the independence of J_1 and J_2. Next write

$$\begin{aligned}
\mathbb{E}(e^{-s_1 D(t_1)} e^{-s_2 D(t_1+t_2)}) &= \mathbb{E}(e^{-s_1 D(t_1)} e^{-s_2[D(t_1)+D(t_1+t_2)-D(t_1)]}) \\
&= \mathbb{E}(e^{-(s_1+s_2)D(t_1)} e^{-s_2[D(t_1+t_2)-D(t_1)]}) \\
&= e^{-t_1(s_1+s_2)^\beta} e^{-t_2 s_2^\beta},
\end{aligned}$$

using the fact that $D(t)$ has independent increments. Then

$$\begin{aligned}
\mathbb{E}(e^{-s_1\tau_1 - s_2\tau_2}) &= \mathbb{E}(e^{-s_1 D(W_1-) - s_2 D([W_1+W_2]-)}) \\
&= \mathbb{E}\left[\mathbb{E}\left(e^{-s_1 D(W_1) - s_2 D(W_1+W_2)} \middle| W_1, W_2\right)\right] \\
&= \mathbb{E}\left[e^{-W_1(s_1+s_2)^\beta} e^{-W_2 s_2^\beta}\right] = \frac{\lambda}{\lambda + (s_1+s_2)^\beta} \cdot \frac{\lambda}{\lambda + s_2^\beta} = \mathbb{E}(e^{-s_1 T_1} e^{-s_2 T_2}).
\end{aligned}$$

Now an application of the continuous mapping theorem shows that $(J_1, \ldots, J_n)$ is identically distributed with $(X_1, \ldots, X_n)$ for any positive integer n. Then (X_n) is an IID sequence, so $N_1(E(t))$ is a renewal process. □

Next we want to show that the FTPP $N_1(E(t))$, and hence also the FPP $N_\beta(t)$, occurs naturally as a CTRW scaling limit. Suppose now that $\mathbb{P}(J_n > t) = t^{-\beta} L(t)$, where $0 < \beta < 1$ and L is slowly varying. For example, this is true of the Mittag-Leffler waiting times. Then J_1 belongs to the strict domain of attraction of some stable law D with index $0 < \beta < 1$, i.e., there exist $b_n > 0$ such that

$$b_n(J_1 + \cdots + J_n) \Rightarrow D, \qquad (2.5)$$

where $D(1) = D > 0$ almost surely, and $\Rightarrow$ denotes convergence in distribution. Let $b(t) = b_{[t]}$. Then $b(t) = t^{-1/\beta} L_0(t)$ for some slowly varying function $L_0(t)$ (e.g., see [14, XVII.5]). Since b varies regularly with index $-1/\beta$, b^{-1} is regularly varying with index $1/\beta > 0$ and so by [36, Property 1.5.5] there exists a regularly varying function $\tilde{b}$ with index β such that $1/b(\tilde{b}(c)) \sim c$, as $c \to \infty$. Here we use the notation $f \sim g$ for positive functions f, g if and only if $f(c)/g(c) \to 1$ as $c \to \infty$. Let $T_n = J_1 + \cdots + J_n$ and define a renewal process

$$R(t) = \max\{n \geq 0 : T_n \leq t\} \tag{2.6}$$

with these waiting times. Next, construct a CTRW with iid Bernoulli jumps $Y_n^{(p)}$ with $\mathbb{P}(Y_n^{(p)} = 1) = p$ and $\mathbb{P}(Y_n^{(p)} = 0) = 1 - p$, independent of (J_n). Let $S^{(p)}(n) = Y_1^{(p)} + \cdots + Y_n^{(p)}$, a binomial random variable. Then $S^{(p)}(R(t))$ is a CTRW with heavy tailed waiting times and Bernoulli jumps.

Theorem 2.3. *The FTPP is the process limit of a CTRW sequence:*

$$\left\{S^{(1/\tilde{b}(c))}([\lambda R(ct)])\right\}_{t\geq 0} \Rightarrow \left\{N_1(E(t))\right\}_{t\geq 0} \tag{2.7}$$

as $c \to \infty$ in the M_1 topology on $D([0,\infty), \mathbb{R})$.

Proof. Since the sequence (J_n) is in the strict domain of attraction of a β-stable random variable D, [26, Corollary 3.4] shows that

$$\left\{\tilde{b}(c)^{-1} R(ct)\right\}_{t\geq 0} \Rightarrow \left\{E(t)\right\}_{t\geq 0} \quad \text{as } c \to \infty.$$

in the Skorokhod J_1 topology, where $D(t)$ is the stable subordinator with $D(1) = D$, and $E(t)$ is given by (1.7).

Since the binomial random variable $S^{(p)}(n)$ has LT $\mathbb{E}(e^{-sS^{(p)}(n)}) = (1 + (e^{-s} - 1)p)^n$ for any $n \geq 0$, it follows that

$$\mathbb{E}(e^{-sS^{(p)}([\lambda t/p])}) = (1 + (e^{-s} - 1)p)^{[\lambda t/p]} \to \exp(-\lambda t(1 - e^{-s})),$$

as $p \to 0$, using the fact that $(1 + ap)^{1/p} \to e^a$ as $p \to 0$. It follows by the continuity theorem for LT that $S^{(p)}([\lambda t/p]) \Rightarrow N_1(t)$ for any $t > 0$, since $\exp(-\lambda t(1 - e^{-s}))$ is the LT of the Poisson random variable $N_1(t)$. Then a standard argument (e.g., see [25, Example 11.2.18] shows that we also get

$$\left\{S^{(p)}([\lambda t/p])\right\}_{t\geq 0} \overset{f.d.}{\Longrightarrow} \left\{N_1(t)\right\}_{t\geq 0},$$

as $p \to 0$, where $\overset{f.d.}{\Longrightarrow}$ denotes convergence of all finite dimensional distributions. Since the sample paths of $S^{(p)}([\lambda t/p])$ are increasing and $N_1(t)$ is continuous in probability, being a Lévy process, J_1 convergence follows using [7, Theorem 3].

Since the CTRW waiting times (J_n) are independent of the jumps $(Y_n^{(p)})$, and since $1/\tilde{b}(c) \to 0$ as $c \to \infty$, it follows that

$$(S^{(1/\tilde{b}(c))}([\lambda t\tilde{b}(c)]), \tilde{b}(c)^{-1} R(ct)) \Rightarrow (N_1(t), E(t))$$

in the J_1 topology of the product space $D([0,\infty), \mathbb{R}\times\mathbb{R})$, by [6, Theorem 3.2]. Since the process $E(t)$ is nondecreasing and continuous, [37, Theorem 13.2.4] along with the continuous mapping theorem yields

$$S^{(1/\tilde{b}(c))}([\lambda R(ct)]) = S^{(1/\tilde{b}(c))}([\lambda \cdot \tilde{b}(c)^{-1} R(ct) \cdot \tilde{b}(c)]) \Rightarrow N_1(E(t))$$

in the M_1 topology on $D([0,\infty), \mathbb{R})$. □

Remark 2.4. For the specific case of Mittag-Leffler waiting times, where $\mathbb{P}(J_n > t) = E_\beta(-t^\beta)$, we can take $b_n = n^{-1/\beta}$ in (2.5). To check this, note that

$$\mathbb{E}(e^{-sb_nT_n}) = \left(\frac{1}{1+(sb_n)^\beta}\right)^n = \left(1 - \frac{s^\beta}{n+s^\beta}\right)^n \to e^{-s^\beta} = \mathbb{E}(e^{-sD(1)})$$

as $n\to\infty$, using the fact that $(1-a_n/n)^n \to e^{-a}$ when $a_n \to a$. Then $\tilde{b}(c) = c^\beta$ and the CTRW convergence (2.7) reduces to $S^{(c^{-\beta})}([\lambda R(ct)]) \Rightarrow N_1(E(t))$ as $c\to\infty$. Substitute $p = c^{-\beta}$ to get

$$S^{(p)}([\lambda R(p^{-1/\beta} t)]) \Rightarrow N_1(E(t)), \quad \text{as } p \to 0. \tag{2.8}$$

Remark 2.5. The proof of Theorem 2.2 uses the fact that, if $D(t)$ is a β-stable subordinator and W_1 is exponential, then $D(W_1)$ has a Mittag-Leffler distribution. This fact was first noticed by Pillai [32], who showed that $W_1^{1/\beta} D(1)$ is Mittag-Leffler. These are equivalent because $D(t)$ is identically distributed with $t^{1/\beta}D(1)$. This Mittag-Leffler distribution is also known as the positive Linnik law, e.g., see Huillet [18]. It has the property of geometric stability: A geometric random sum of Mittag-Leffler variables is again Mittag-Leffler, e.g., see Kozubowski [20].

3. Fractional calculus

This section develops some interesting connections between the fractional Poisson process and fractional calculus. In the process, some apparent inconsistencies in the existing literature will be explained. Behgin and Orsingher [5, Eq. (2.17)] show that the FPP of order $0<\beta<1$ has distribution

$$\mathbb{P}(N_\beta(t) = k) = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} V(x,t)\, dx, \tag{3.1}$$

where $V(x,t)$ is a "folded PDF" defined on $x>0$, for each $t>0$, by $V(x,t) = 2v(x,t)$, and $v(x,t)$ is another PDF with $x\in\mathbb{R}$ for each $t>0$ that solves

$$\begin{aligned} \partial_t^{2\beta} v(x,t) &= \partial_x^2 v(x,t); \\ v(x,0) &= \delta(x); \\ \partial_t v(x,0) &\equiv 0, \quad \text{if } 1/2<\beta<1. \end{aligned} \tag{3.2}$$

It is also stated in [5, Eq. (1.9)] that the FPP $N_\beta(t) = N_1(T_t)$, where T_t is a random process with PDF $V(x,t)$ for $t>0$. However, that process is identified only in terms of its one dimensional distributions (PDF). Theorem 2.2 shows that the inverse stable subordinator $E(t)$ is one such process.

On the other hand, a simple conditioning argument shows that the equivalent FTPP process has distribution

$$\mathbb{P}(N_1(E(t)) = k) = \int_0^\infty P(N_1(x) = k) h(x,t)\, dx = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} h(x,t)\, dx \tag{3.3}$$

where $h(x, t)$ is the density of $E(t)$, a PDF on $x > 0$ for each $t > 0$. It follows from [28, Theorem 4.1] that this PDF solves

$$\partial_t^\beta h(x,t) = -\partial_x h(x,t); \quad h(x,0) = \delta(x). \tag{3.4}$$

In view of Theorem 2.2, the two distributions (3.1) and (3.3) must be equal. Thus, the main purpose of this section is to reconcile the two fractional differential equations (3.2) and (3.4).

Theorem 3.1. *Let $N_\beta(t)$ be a fractional Poisson process* (1.3) *with $0 < \beta < 1$, so that* (3.1) *holds. Let $N_1(E(t))$ be the equivalent fractal time Poisson process, where $E(t)$ is the standard inverse β-stable subordinator with probability density function $h(x, t)$, so that* (3.3) *holds. Then*

$$h(x,t) = 2v(x,t) \quad \textit{for all } x > 0 \textit{ and } t > 0. \tag{3.5}$$

In particular, the two fractional partial differential equations (3.2) *and* (3.4) *are consistent, in the sense that the folded solution $V(x, t) = 2v(x, t)$ to* (3.2) *coincides with the solution $h(x, t)$ to* (3.4).

Proof. Mainardi [23, Eq. (3.2)] shows that the solution to the fractional diffusion-wave equation (3.2) has LT

$$\tilde{v}(x,s) = \int_0^\infty e^{-st} v(x,t) dt = \frac{1}{2} s^{\beta-1} e^{-|x| s^\beta} \tag{3.6}$$

while [28, Eq. (3.13)] shows that

$$\tilde{h}(x,s) = s^{\beta-1} e^{-x s^\beta}. \tag{3.7}$$

Since both are differentiable in t, they are also continuous, so LT uniqueness for continuous functions implies (3.5).

Take Fourier transforms in (3.6) to see that the solution to (3.2) has Fourier-Laplace transform (FLT)

$$\bar{v}(k,s) = \int_0^\infty e^{-st} \int_{-\infty}^\infty e^{-ikx} v(x,t) dx dt = \frac{s^{2\beta-1}}{s^{2\beta} + k^2}, \tag{3.8}$$

where we have used the fact that $e^{-a|x|}$ has FT $2a/(a^2 + k^2)$. Rearrange to get

$$s^{2\beta} \bar{v}(k,s) - s^{2\beta-1} = -k^2 \bar{v}(k,s)$$

and invert the FT to get

$$s^{2\beta}\tilde{v}(x,s) - s^{2\beta-1}v(x,0) = \partial_x^2\tilde{v}(x,s),$$

using the fact that $\partial_x f(x)$ has FT $(ik)\hat{f}(k)$ and $v(x,0) = \delta(x)$ has FT $\hat{v}(k,0) \equiv 1$. To invert the LT, note that the Caputo fractional derivative $\partial_t^\beta f(t)$ has LT $s^\beta \tilde{f}(s) - s^{\beta-1}f(0)$ if $0 < \beta \leq 1$, and LT $s^\beta\tilde{f}(s) - s^{\beta-1}f(0) - s^{\beta-2}f'(0)$ if $1 < \beta \leq 2$. This is easy to verify from the definition (1.5), using the corresponding formula for the integer derivative, along with the fact that $s^{\beta-1}$ is the LT of $t^{-\beta}/\Gamma(1-\beta)$. Now use the remaining initial condition $\partial_t v(x,0) \equiv 0$ for $1/2 < \beta < 1$ to invert the LT, and arrive at (3.2).

Likewise, the solution to (3.4) has FLT

$$\bar{h}(k,s) = \int_0^\infty e^{-ikx}s^{\beta-1}e^{-xs^\beta}dx = \frac{s^{\beta-1}}{s^\beta + ik}, \tag{3.9}$$

using the fact that $e^{ax}I(x \geq 0)$ has FT $1/(a+ik)$. To see that these are consistent, compute the FLT of $h(|x|,t)$:

$$\begin{aligned}\int_0^\infty e^{-st}\int_{-\infty}^\infty e^{-ikx}h(|x|,t)dxdt &= \int_0^\infty e^{-st}\left(\int_0^\infty e^{-ikx}h(x,t)dx + \int_0^\infty e^{ikx}h(x,t)dx\right)dt\\ &= \frac{s^{\beta-1}}{s^\beta+ik} + \frac{s^{\beta-1}}{s^\beta-ik} = 2\left(\frac{s^{2\beta-1}}{s^{2\beta}+k^2}\right) = 2\bar{v}(k,s).\end{aligned}$$

Invert the FLT to see that $h(|x|,t) = 2v(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$. To verify the LFT solution, take FT in (3.4) to get

$$\partial_t^\beta \hat{h}(k,t) = -ik\,\hat{h}(x,t)$$

and apply the LT to get $s^\beta\bar{h}(k,s) - s^{\beta-1} = -ik\,\bar{h}(k,s)$, using the point source initial condition $\hat{h}(k,0) \equiv 1$. □

Remark 3.2. Behgin and Orsingher [5, Eq. (2.21)] show that

$$v(x,t) = \frac{1}{2\Gamma(1-\beta)}\int_0^t (t-w)^{-\beta}p(|x|,t)dx$$

where $p(x,t)$ is the density of the stable subordinator $D(t)$, while [28, Theorem 3.1] implies that

$$h(x,t) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-w)^{-\beta}p(x,t)dx.$$

This gives another proof that $h(|x|,t) = 2v(x,t)$.

The equivalence in Theorem 3.1 results from folding the solution to the fractional diffusion-wave equation (3.2). Another fractional partial differential equation for the density $h(x,t)$ of the standard inverse β-stable subordinator $E(t)$, which is closer to the form (3.2), can be obtained by arguments similar to those used in [3] to connect the inverse stable subordinator to iterated Brownian motion. In that theory, it is customary to avoid distributions by imposing a functional initial condition.

Theorem 3.3. *Let $E(t)$ be the standard inverse β-stable subordinator with density $h(x,t)$. Then for any $f \in L_2(\mathbb{R}) \cap C^1(\mathbb{R})$, the function*

$$u(x,t) = \mathbb{E}_x[f(E(t))] = \int_0^\infty f(x+y)h(y,t)dy \tag{3.10}$$

solves the fractional differential equation

$$\partial_t^{2\beta} u(x,t) = -\partial_x f(x) \frac{t^{\beta-1}}{\Gamma(1-\beta)} + \partial_x^2 u(x,t); \quad u(x,0) = f(x). \tag{3.11}$$

In particular, when $\beta = 1/2$, (3.10) *solves*

$$\partial_t u(x,t) = \frac{-\partial_x f(x)}{\sqrt{\pi t}} + \partial_x^2 u(x,t); \quad u(0,x) = f(x), \tag{3.12}$$

and in this case we also have $u(x,t) = \mathbb{E}_x[f(|B(t)|)]$, where $B(t)$ is a Brownian motion with variance $2t$.

Proof. From (3.9), we have

$$\bar{u}(k,s) = \frac{s^{\beta-1}\hat{f}(k)}{s^\beta + ik} = \frac{s^{\beta-1}\hat{f}(k)}{s^\beta + ik} \cdot \frac{s^\beta - ik}{s^\beta - ik} = \frac{s^{2\beta-1} - iks^{\beta-1}}{s^{2\beta} + k^2} \hat{f}(k)$$

so that $s^{2\beta}\bar{u}(k,s) - s^{2\beta-1}\hat{f}(k) = -ik\hat{f}(k)s^{\beta-1} - k^2\bar{u}(k,s)$, which inverts to (3.11). It is well known that the Brownian motion first passage time $D(y) = \inf\{t > 0 : B(t) > y\}$ is a stable subordinator with index $\beta = 1/2$ [1, Example 1.3.19]. Then it is easy to see that

$$E(t) = \inf\{y > 0 : D(y) > t\} = \sup\{B(r) : 0 \le r \le t\}$$

and this recovers the fact, typically proven using the reflection principle, that

$$P(E(t) > y) = 2P(B(t) > y).$$

Then $E(t)$ and $|B(t)|$ have the same one dimensional distributions, so we also have $u(x,t) = \mathbb{E}_x[f(|B(t)|)]$. Note that $X(t) = X(0) - t$ is *a fortiori* a continuous Markov process associated with the shift semigroup $T(t)f(x) = \mathbb{E}_x[f(X(t))] = f(x-t)$ with generator

$$L_x f(x) = \lim_{t \to 0+} \frac{T(t)f(x) - f(x)}{t} = -\partial_x f(x).$$

Then [3, Corollary 3.4] implies that $u(x,t)$ solves the equation

$$\partial_t u(x,t) = \frac{L_x f(x)}{\sqrt{\pi t}} + L_x{}^2 u(x,t); \quad u(0,x) = f(x).$$

When $L_x = -\partial_x$, this reduces to (3.12), a special case of (3.11) with $\Gamma(1/2) = \sqrt{\pi}$. □

Remark 3.4. In the case $f(x) = \delta(x)$, Theorem 3.3 gives an alternative governing equation for $h(x,t)$. Note that (3.11) is very similar to the governing equation (3.2) for the unfolded PDF.

Remark 3.5. The process $|B(t)|$ in Theorem 3.3 is not the same process as the inverse 1/2-stable subordinator $E(t)$ in Theorem 2.2, although they have the same one dimensional distributions. Hence, the FTPP is not the same as the Brownian time Poisson process $N_1(|B(t)|)$. However, we do have $E(t) = \sup\{B(r) : 0 \le r \le t\}$, so that a Poisson process subordinated to the supremum of a Brownian motion is an FPP with $\beta = 1/2$.

Remark 3.6. Let $E(t)$ be the standard inverse stable subordinator of index $\beta = 1/m$ for integer $m > 1$. Then [3, Remark 3.11], [31, Theorem 1.1] and Keyantuo and Lizama [19, Theorem 3.3] imply that $u(x,t) = \mathbb{E}_x[f(E(t))]$ solves

$$\partial_t u(x,t) = \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)}(-\partial_x)^j f(x) + (-\partial_x)^m u(x,t); \quad u(0,x) = f(x),$$

for $t > 0$ and $x \in \mathbb{R}$, which is then equivalent to (3.4). The proof is similar to Theorem 3.3. For example, when $\beta = 1/3$ use

$$\bar{u}(s,k) = \frac{s^{-2/3}\hat{f}(k)}{s^{1/3}+ik} \cdot \frac{s^{2/3} - s^{1/3}ik + k^2}{s^{2/3} - s^{1/3}ik + k^2} = \frac{1 - s^{-1/3}ik + s^{-2/3}k^2}{s + ik^3}\hat{f}(k).$$

4. Renewal processes and inverse subordinators

Theorem 2.2 shows that a Poisson process, time-changed by an inverse stable subordinator, yields a renewal process with Mittag-Leffler waiting times. This section extends that result to arbitrary subordinators that are strictly increasing. Let $D(t)$ be a strictly increasing Lévy process (subordinator) with $\mathbb{E}[e^{-sD(t)}] = e^{-t\psi_D(s)}$, where the Laplace exponent

$$\psi_D(s) = bs + \int_0^\infty (e^{-sx} - 1)\phi_D(dx), \tag{4.1}$$

$b \ge 0$, and ϕ_D is the Lévy measure of D. Then we must have either

$$\phi_D(0,\infty) = \infty, \tag{4.2}$$

or $b > 0$, or both. Let $E(t)$ be the inverse subordinator (1.7), and recall that $N_1(t)$ is a Poisson process with rate λ.

Theorem 4.1. *The time-changed Poisson process $N_1(E(t))$ is a renewal process whose IID waiting times (J_n) satisfy*

$$\mathbb{P}(J_n > t) = \mathbb{E}[e^{-\lambda E(t)}]. \tag{4.3}$$

Proof. The proof is similar to Theorem 2.2. Take $N_1(t) = \max\{n \ge 0 : V_n \le t\}$, where $V_n = W_1 + \cdots + W_n$, with W_n IID as $\mathbb{P}(W_n > t) = e^{-\lambda t}$. Let

$$\tau_n = \sup\{t > 0 : N_1(E(t)) < n\} = \sup\{t > 0 : E(t) < V_n\}$$

and apply Lemma 2.1 to get $\tau_n = D(V_n-)$. Then, as in the proof of Theorem 2.2, we have

$$\begin{aligned} \mathbb{E}(e^{-s\tau_1}) &= \mathbb{E}(e^{-sD(W_1-)}) \\ &= \mathbb{E}\left[\mathbb{E}\left(e^{-sD(W_1)}\middle|W_1\right)\right] = \mathbb{E}\left[e^{-W_1\psi_D(s)}\right] = \frac{\lambda}{\lambda+\psi_D(s)}. \end{aligned} \tag{4.4}$$

By [28, Corollary 3.5], the IID random variables J_n in (4.3) satisfy

$$\int_0^\infty e^{-st}\,\mathbb{P}(J_n > t)\,dt = \int_0^\infty e^{-st}\,\mathbb{E}[e^{-\lambda E(t)}]\,dt = \frac{\psi_D(s)}{s(\lambda+\psi_D(s))}. \tag{4.5}$$

Integrate by parts to get

$$\int_0^\infty e^{-st}\,\mathbb{P}_{J_n}(dt) = \int_0^\infty se^{-st}\,\left[1-\mathbb{P}(J_n > t)\right]\,dt = 1 - \frac{\psi_D(s)}{\lambda+\psi_D(s)} = \frac{\lambda}{\lambda+\psi_D(s)}, \tag{4.6}$$

which shows that $T_1 = J_1$ is identically distributed with τ_1. Extend this argument, as in the proof of Theorem 2.2, to show that $(T_1, \ldots, T_n)$ is identically distributed with $(\tau_1, \ldots, \tau_n)$ for any positive integer n. For example, when $n = 2$, write

$$\mathbb{E}(e^{-s_1D(t_1)}e^{-s_2D(t_1+t_2)}) = \mathbb{E}(e^{-(s_1+s_2)D(t_1)}e^{-s_2[D(t_1+t_2)-D(t_1)]}) = e^{-t_1\psi_D(s_1+s_2)}e^{-t_2\psi_D(s_2)}$$

and condition to get

$$\begin{aligned} \mathbb{E}(e^{-s_1\tau_1-s_2\tau_2}) &= \mathbb{E}(e^{-s_1D(W_1-)-s_2D([W_1+W_2]-)}) \\ &= \mathbb{E}\left[\mathbb{E}\left(e^{-s_1D(W_1)-s_2D(W_1+W_2)}\middle|W_1,W_2\right)\right] \\ &= \mathbb{E}\left[e^{-W_1\psi_D(s_1+s_2)}e^{-W_2\psi_D(s_2)}\right] = \frac{\lambda}{\lambda+\psi_D(s_1+s_2)}\cdot\frac{\lambda}{\lambda+\psi_D(s_2)}. \end{aligned}$$

On the other hand,

$$\mathbb{E}(e^{-s_1T_1}e^{-s_2T_2}) = \mathbb{E}(e^{-s_1J_1}e^{-s_2(J_1+J_2)}) = \frac{\lambda}{\lambda+\psi_D(s_1+s_2)}\cdot\frac{\lambda}{\lambda+\psi_D(s_2)}$$

using the fact that (J_n) are IID. To finish the proof, use continuous mapping to show that $(J_1, \ldots, J_n)$ is identically distributed with $(X_1, \ldots, X_n)$, where $X_n = \tau_n - \tau_{n-1}$ are the waiting times between jumps for the process $N_1(E(t))$. □

Remark 4.2. (i) Let $N_D(t)$ denote the renewal process from Theorem 4.1, so that

$$N_D(t) = \max\{n \ge 0 : T_n \le t\}, \tag{4.7}$$

where $T_n = \sum_{i=1}^n J_i$ and (J_n) are IID according to (4.3). Theorem 4.1 shows that $N_D(t) = N_1(E(t))$. This extends the relation $N_\beta(t) = N_1(E(t))$ from Theorem 2.2,

the special case of an inverse stable subordinator $E(t)$ and Mittag-Leffler waiting times J_n, to a general inverse subordinator.
(ii) Let $M(t) = \mathbb{E}(N_D(t))$ denote the renewal function of the renewal process $N_D(t)$. Then using Lageras [21, Equation 4], it follows that the LT of $M(t)$ is $\lambda/\psi_D(s)$.

5. CTRW scaling limits and governing equations

In this section, we extend the fractional calculus results of Section 3 to the inverse subordinators of Section 4. A general theory of CTRW scaling limits and governing equations is developed in [28]. Consider a sequence of CTRW indexed by a scale parameter $c > 0$. Take J_n^c nonnegative IID random variables representing the waiting times between particle jumps and $T^c(n) = \sum_{i=1}^n J_i^c$, the time of the nth jump. Let Y_i^c be IID random vectors on $\mathbb{R}^d$ representing the particle jumps, independent of the waiting times, and set $S^c(n) = \sum_{i=1}^n Y_i^c$, the location of the particle after n jumps. Define $N_t^c = \max\{n \geq 0 : T^c(n) \leq t\}$, the number of jumps by time $t \geq 0$ and

$$X^c(t) = S^c(N_t^c) = \sum_{i=1}^{N_t^c} Y_i^c \tag{5.1}$$

the position of the particle at time $t \geq 0$ and scale $c > 0$. Assume a triangular array limit

$$\{(S^c(ct), T^c(ct))\}_{t\geq 0} \Rightarrow \{(A(t), D(t)\}_{t\geq 0}, \quad \text{as } c \to \infty, \tag{5.2}$$

in the J_1 topology on $D([0,\infty), \mathbb{R}^d \times \mathbb{R}_+)$, so that $A(t)$ and $D(t)$ are independent Lévy processes on $\mathbb{R}^d$ and $\mathbb{R}$, respectively. Since the waiting times are nonnegative, $D(t)$ is a subordinator. In this section, we assume the drift $b = 0$ in (4.1), as well as condition (4.2) and

$$\int_0^1 y |\ln y| \, \phi_D(dy) < \infty. \tag{5.3}$$

Assumption (4.2) implies that the process $\{D(t)\}$ is strictly increasing, i.e., $D(t)$ is not compound Poisson. Then [28, Theorem 3.1] shows that the inverse subordinator $E(t)$ in (1.7) has a Lebesgue density

$$h(x,t) = \int_0^t \phi_D(t-y,\infty)\, \mathbb{P}_{D(x)}(dy). \tag{5.4}$$

Write $\mathbb{E}[e^{-sD(t)}] = e^{-t\psi_D(s)}$, as before. Let $P(x,t) = \mathbb{P}(A(t) \leq x)$ be the distribution function of $A(t)$, and write

$$\hat{P}(k,t) = \int e^{-ik\cdot x} P(dx,t) = e^{-t\psi_A(k)},$$

where $\psi_A(k)$ is the Fourier symbol of A. The symbols define pseudo-differential operators: $\psi_D(\partial_t) f(t)$ has LT $\psi_D(s)\tilde{f}(s)$, and $\psi_A(-iD_x)f(x)$ has FT $\psi_A(k)\hat{f}(k)$, for suitable functions f. Then [28, Theorem 2.1] establishes the CTRW scaling limit

$$\{X^c(t)\}_{t\geq 0} \Rightarrow \{A(E(t))\}_{t\geq 0}, \quad \text{as } c\to\infty, \tag{5.5}$$

in the M_1-topology on $D([0,\infty),\mathbb{R}^d)$. Recall that a function Q is a *mild solution* to a space-time pseudo-differential equation if its (Fourier-Laplace or Laplace-Laplace) transform solves the equivalent algebraic equation in transform space. The next result is a small extension of [28, Theorem 4.1].

Theorem 5.1. *Assume* (5.2) *holds, where $D(t)$ is a subordinator without drift such that conditions* (4.2) *and* (5.3) *hold. The distribution function of the CTRW limit process $A(E(t))$ in* (5.5) *is given by*

$$Q(x,t) = \int_0^\infty P(x,u)h(u,t)\,du \tag{5.6}$$

where $h(u,t)$ is the density (5.4) *of the inverse subordinator $E(t)$. The distribution function $Q(x,t)$ solves the generalized Cauchy problem*

$$\psi_D(\partial_t)Q(x,t) = -\psi_A(-iD_x)Q(x,t) + H(x)\phi_D(t,\infty) \tag{5.7}$$

in the mild sense, where $H(x) = I(x\geq 0)$ is the Heaviside function. Furthermore, $P(x,u)$ solves the Cauchy problem

$$\partial_t P(x,t) = -\psi_A(-iD_x)P(x,t); \quad P(x,0) = H(x), \tag{5.8}$$

and $h(x,t)$ solves the inhomogeneous Cauchy problem

$$\partial_x h(x,t) = -\psi_D(\partial_t)h(x,t) + \delta(x)\phi_D(t,\infty). \tag{5.9}$$

Proof. The proof is similar to [28, Theorem 4.1]. Equation (5.6) follows from a simple conditioning argument. Apply [28, Theorem 3.6] to see that $Q(x,t)$ has FLT

$$\bar{Q}(k,s) = \int_0^\infty e^{-st}\int_{\mathbb{R}^d} e^{-ik\cdot x}Q(dx,t)\,dt = \frac{1}{s}\frac{\psi_D(s)}{\psi_A(k)+\psi_D(s)} \tag{5.10}$$

and rearrange to get

$$\psi_D(s)\,\bar{Q}(k,s) = -\psi_A(k)\,\bar{Q}(k,s) + s^{-1}\psi_D(s). \tag{5.11}$$

From [28, Eq. (3.12)] we get

$$\int_0^\infty e^{-su}\phi_D(u,\infty)\,du = s^{-1}\psi_D(s). \tag{5.12}$$

Now invert the FLT (5.11), using (5.12) and $\int e^{-ik\cdot x}H(dx)\equiv 1$, to arrive at (5.7). It is well known that $P(x,t)$ solves the Cauchy problem (5.8), see for example [17]. Equation (4.5) shows that the bivariate Laplace transform (LLT)

$$\tilde{h}(\lambda,s) = \int_0^\infty\int_0^\infty e^{-\lambda z - st}h(z,t)\,dt\,dz = \frac{1}{s}\frac{\psi_D(s)}{\lambda+\psi_D(s)}.$$

This rearranges to

$$\lambda\tilde{h}(\xi,s) = -\psi_D(s)\tilde{h}(\lambda,s) + s^{-1}\psi_D(s).$$

Inverting the LLT using (5.12) to see that $h(x,t)$ solves (5.9). $\square$

For any random walk $S(n) = \sum_{i=1}^n Y_i$, the compound Poisson process $A(t) = S(N_1(t))$ is a Lévy process. Introduce IID waiting times (4.3) between these random walk jumps to get a CTRW. In this case, the CTRW is exactly of the form $A(E(t))$, without passing to the limit. Then the governing equations in Theorem 5.1 pertain to the CTRW itself.

Theorem 5.2. *Assume $D(t)$ is a subordinator without drift such that conditions* (4.2) *and* (5.3) *hold, and let $E(t)$ be the inverse subordinator* (1.7)*. Take J_n IID waiting times according to* (4.3)*, and let $N_D(t)$ denote the renewal process* (4.7)*. Take Y_n IID jumps on $\mathbb{R}^d$, independent from (J_n), with common distribution μ, and let $S(n) = \sum_{i=1}^n Y_i$. Then the distribution function $P(x,t) = \mathbb{P}(X(t) \le x)$ of the CTRW $X(t) = S(N_D(t))$ solves the generalized Cauchy problem*

$$\psi_D(\partial_t)P(x,t) = -\lambda P(x,t) + \lambda \int P(x-y,t)\,\mu(dy) + H(x)\phi_D(t,\infty) \tag{5.13}$$

in the mild sense. Furthermore, $X(t) = A(E(t))$, where $A(t) = S(N_1(t))$ is a compound Poisson process.

Proof. Theorem 4.1 yields $N_D(t) = N_1(E(t))$, and then the CTRW is

$$X(t) = S(N_D(t)) = S(N_1(E(t))) = A(E(t)).$$

A standard conditioning argument shows that the compound Poisson FT $\hat{P}(k,t) = e^{-t\psi_A(k)}$, where the Fourier symbol $\psi_A(k) = \lambda(1-\hat{\mu}(k))$. The inverse FT of $\psi_A(k)\hat{f}(k)$ is

$$\psi_A(-iD_x)f(x) = -\lambda f(x) + \lambda \int f(x-y)\,\mu(dy) \tag{5.14}$$

using the FT convolution property. Now Theorem 5.1 implies that (5.13) holds. $\square$

Remark 5.3. In the situation of Theorem 5.2, where $A(t)$ is compound Poisson, the distribution function $P(x,t) = \mathbb{P}(A(t) \le x)$ solves the Cauchy problem (5.8), which can be written in this case as

$$\partial_t P(x,t) = -\lambda P(x,t) + \lambda \int_{-\infty}^{\infty} P(x-y,t)\,\mu(dy); \quad P(x,0) = H(x). \tag{5.15}$$

This is the Kolmogorov forward equation for the Markov process $A(t)$. If μ has density $w(x)$, apply ∂_x on both sides of (5.15) to see that the probability density $p(x,t) = \partial_x P(x,t)$ of $A(t)$ solves (1.6). If D is the stable subordinator with Laplace symbol $\psi_D(s) = s^\beta$, then (5.13) holds with $\phi_D(t,\infty) = t^{-\beta}/\Gamma(1-\beta)$ and $\psi_D(\partial_t) = \mathbb{D}_t^\beta$, the Riemann-Liouville fractional derivative. The Riemann-Liouville fractional derivative is defined for $0 \le n-1 < \beta < n$ by

$$\mathbb{D}_t^\beta g(t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_0^t (t-r)^{n-1-\beta} g^{(n)}(r)\,dr, \tag{5.16}$$

which differs from the Caputo derivative (1.5) in that the derivative is applied after the integration. The LT of $\mathbb{D}_t^\beta g(t)$ is $s^\beta \tilde{g}(s)$. Apply ∂_x to both sides of (5.13) in this case to get

$$\mathbb{D}_t^\beta p(x,t) = -\lambda p(x,t) + \lambda \int p(x-y,t)\,\mu(dy) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)},$$

the fractional kinetic equation of Zaslavsky [38]. To recover (1.4), use $\partial_t^\beta g(t) = \mathbb{D}_t^\beta g(t) - g(0)t^{-\beta}/\Gamma(1-\beta)$ and $p(x,0) = \delta(x)$.

Remark 5.4. In the special case where $\mu = \varepsilon_1$ is a point mass, so that $Y_n = 1$ almost surely, $A(t) = N_1(t)$ is a Poisson process with rate $\lambda > 0$. Then the distribution function $P(x,t)$ of the renewal process $N_D(t) = A(E(t))$ solves

$$\psi_D(\partial_t)P(x,t) = -\lambda[P(x,t) - P(x-1,t)] + H(x)\phi_D(t,\infty). \tag{5.17}$$

If D is the stable subordinator with Laplace symbol $\psi_D(s) = s^\beta$, this reduces to

$$\partial_t^\beta P(x,t) = -\lambda[P(x,t) - P(x-1,t)]$$

as in Remark 5.3. The probability mass function $p(n,t) = P(n,1) - P(n-1,t) = \Delta P(n,t)$ for $n > 0$. Apply the difference operator Δ on both sides to obtain

$$\partial_t^\beta p(n,t) = -\lambda[p(n,t) - p(n-1,t)]$$

which is Eq. (1.1) in Behgin and Orsingher [5].

Remark 5.5. Scher and Lax [35] showed that a CTRW with waiting time distribution ω and jump distribution ν has FLT

$$\bar{Q}(k,s) = \frac{1}{s}\frac{1-\tilde{\omega}(s)}{1-\tilde{\omega}(s)\hat{\nu}(k)}$$

, where $\hat{\nu}(k) = \int e^{-ik\cdot x}\nu(dx)$. To reconcile with Theorem 5.2, recall from (4.6) that the waiting times (4.3) in Theorem 5.2 have LT

$$\tilde{\omega}(s) = \int e^{-st}\omega(dt) = \frac{\lambda}{\lambda + \psi_D(s)}$$

and then it follows that $\psi_D(s) = \lambda(1-\tilde{\omega}(s))/\tilde{\omega}(s)$. The jumps Y_n in Theorem 5.2 have Fourier symbol $\psi_A(k) = \lambda(1-\hat{\mu}(k))$ and then (5.10) implies

$$\bar{Q}(k,s) = \frac{1}{s}\frac{\psi_D(s)}{\psi_A(k)+\psi_D(s)} = \frac{1}{s}\frac{\frac{1-\tilde{\omega}(s)}{\tilde{\omega}(s)}}{\frac{1-\tilde{\omega}(s)}{\tilde{\omega}(s)} + (1-\hat{\mu}(k))} = \frac{1}{s}\frac{1-\tilde{\omega}(s)}{1-\tilde{\omega}(s)\hat{\mu}(k)}$$

which provides a different proof that the CTRW equals $A(E(t))$ in this case. To simulate the sample paths of the non-Markovian process $A(E(t))$, it is sufficient to simulate the CTRW. In particular, the renewal process $N_D(t)$ gives the exact jump times of the inverse subordinator $E(t)$.

Remark 5.6. In the general case, where $A(t)$ is not compound Poisson, Theorem 5.2 provides a useful approximation. Given a Lévy process $A(t)$, take $Y_n = A(n) - A(n-1)$, so that $S(n) = A(n)$. Take $N(t)$ a Poisson process with rate 1, so that $S(\lambda^{-1}N(\lambda t))$ is compound Poisson with Fourier symbol

$$\lambda(1 - e^{-\lambda^{-1}\psi_A(k)}) \to \psi_A(k), \quad \text{as } \lambda \to \infty.$$

Then $S(\lambda^{-1}N(\lambda t)) \Rightarrow A(t)$ as $\lambda \to \infty$, and the CTRW with IID waiting times (4.3) and these compound Poisson jumps converges to $A(E(t))$ as $\lambda \to \infty$. As in Remark 5.5, this fact can be used to simulate sample paths of the process $A(E(t))$. This fact has been exploited by Fulger, Scalas and Germano [15] to develop fast simulation methods for space-time fractional diffusion equations.

Example 5.7. Tempered stable subordinators are theoretically interesting [4, 33] and practically useful [13, 29]. Take $D(t)$ tempered stable with Laplace symbol $\psi_D(s) = (s+a)^\beta - a^\beta$ for $a > 0$ and $0 < \beta < 1$, and let $E(t)$ be its inverse (1.7). Theorem 4.1 shows that $N_1(E(t))$ is a renewal process. Let (τ_n) denote the arrival times of this renewal process, and use (4.4) to get

$$\mathbb{E}(e^{-s\tau_1}) = \frac{\lambda}{\lambda + (s+a)^\beta - a^\beta}.$$

This tempered fractional Poisson process $N_1(E(t))$ has tempered Mittag-Leffler waiting times, but with a different rate parameter: Use (2.4) to see that the Mittag-Leffler PDF $f(t) = \partial_t[1 - E_\beta(-\eta t^\beta)]$ has Laplace transform $\eta/(\eta + s^\beta)$, and so

$$\int_0^\infty e^{-st} f(t) e^{-at} dt = \frac{\eta}{\eta + (s+a)^\beta}.$$

Of course $f(t)e^{-at}$ is not a PDF, and in fact we have (set $s = 0$ above)

$$\int_0^\infty f(t) e^{-at} dt = \frac{\eta}{\eta + a^\beta}.$$

Then the tempered Mittag-Leffler PDF $f_a(t) = f(t)e^{-at}(\eta + a^\beta)/\eta$ has LT

$$\int_0^\infty e^{-st} f_a(t) dt = \frac{\eta + a^\beta}{\eta + (s+a)^\beta} = \frac{\lambda}{\lambda + (s+a)^\beta - a^\beta} = \mathbb{E}(e^{-s\tau_1})$$

when $\eta + a^\beta = \lambda$. Cartea and del Castillo-Negrete [9] show that the tempered fractional derivative $\psi_D(\partial_t)g(t) = e^{-at}\,\partial_t^\beta[e^{at}\,g(t)] - a^\beta g(t)$. It is also known (e.g., see [4]) that the corresponding Lévy measure is exponentially tempered: $\psi_D(dt) = e^{-at}\psi(dt)$, where $\psi(t,\infty) = t^{-\beta}/\Gamma(1-\beta)$ is the Lévy measure of the standard β-stable subordinator. Then Theorem 5.2 shows that the CTRW with tempered Mittag-Leffler waiting times and compound Poisson jumps solves a tempered fractional Cauchy problem

$$e^{-at}\,\partial_t^\beta[e^{at}\,P(x,t)] - a^\beta P(x,t) = \psi_A(-iD_x)P(x,t) + H(x)\phi_D(t,\infty)$$

with $\psi_A(-iD_x)$ given by (5.14) and $\phi_D(t,\infty) = \beta \int_t^\infty e^{-at} t^{-\beta-1} dt/\Gamma(1-\beta)$. More generally, Theorem 5.1 shows that the distribution function of the CTRW scaling limit $A(E(t))$ is governed by this equation, with the corresponding operator $\psi_A(-iD_x)$. Apply ∂_x on both sides of (5.17) to see that the PDF of the renewal process with tempered Mittag-Leffler waiting times solves

$$e^{-at}\,\partial_t^\beta[e^{at}\,p(x,t)] - a^\beta p(x,t) = -\lambda[p(x,t) - p(x-1,t)] + \delta(x)\phi_D(t,\infty).$$

A wide variety of tempered stable models in $\mathbb{R}^d$ are discussed in Rosiński [33]. Random walks in $\mathbb{R}^d$ with tempered stable scaling limit are developed in [10]. For exponentially tempered stable waiting times in $\mathbb{R}^1$, a renewal process with tempered Mittag-Leffler waiting times gives the same process exactly, without taking limits. This can be useful for simulating sample paths.

Example 5.8. Chechkin et al. [12, 11] used distributed order fractional derivatives to model multi-scale anomalous subdiffusion, where a different power law pertains at short and long time scales, and ultraslow diffusion, for a plume of particles spreading at a logarithmic rate. Given a finite Borel measure ν on $(0,1)$, the distributed order fractional derivative is defined by

$$\mathbb{D}_t^\nu g(t) = \int_0^1 \partial_t^\beta g(t)\nu(d\beta), \tag{5.18}$$

where ∂_t^β is the Caputo fractional derivative (1.5). If ν is discrete, this is a linear combination of fractional derivatives. Let $D(t)$ be the distributed order stable subordinator with Laplace symbol $\psi_D(s) = \int s^\beta \nu(d\beta)$ and $E(t)$ its inverse (1.7). If $\nu(d\beta) = p(\beta)d\beta$, where $p(\beta)$ is regularly varying at $\beta = 0$ with index $\alpha - 1$ for some $\alpha > 0$, then $\psi_D(s) = R(\log s)$ and R is regularly varying at infinity with index $-\alpha$, see [27, Lemma 3.1]. Then $E(t)$ is "ultraslow" in that $\mathbb{E}(E(t)^\gamma) = S(\log t)$, where S varies regularly with index $\gamma\alpha$ at infinity, by [27, Theorem 3.9]. Take an IID sequence of mixing variables (B_i) with distribution μ concentrated on $(0,1)$, and assume $\mathbb{P}(J_i^c > u | B_i = \beta) = c^{-1}u^{-\beta}$ for $u \geq c^{-1/\beta}$, so that the waiting times are conditionally Pareto. Then [27, Theorem 3.4] implies that the distributed order stable subordinator is a random walk limit $\sum_{i=1}^{[ct]} J_i^c \Rightarrow D(t)$. This requires $\int (1-\beta)^{-1}\mu(d\beta) < \infty$ so that $\nu(d\beta) = \Gamma(1-\beta)\mu(d\beta)$ is a finite measure. An easy computation shows that the Lévy measure $\phi_D(t,\infty) = \int_0^1 t^{-\beta}\nu(d\beta)/\Gamma(1-\beta)$. Then Theorem 5.1 implies that a CTRW with these conditionally Pareto waiting times has a scaling limit $A(E(t))$ whose distribution $Q(x,t)$ solves the distributed-order fractional diffusion equation

$$\mathbb{D}_t^\nu Q(x,t) = -\psi_A(-iD_x)Q(x,t).$$

If $A(t)$ is compound Poisson, Theorem 5.2 shows that the distribution function $P(x,t)$ of a CTRW with waiting times (4.3) solves

$$\mathbb{D}_t^\nu P(x,t) = -\lambda P(x,t) + \lambda \int P(x-y,t)\,\mu(dy),$$

without passing to the limit. Then the PDF $p(x,t)$ of the renewal process with waiting times (4.3) solves

$$\mathbb{D}_t^{\nu} p(x,t) = -\lambda[p(x,t) - p(x-1,t)].$$

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