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Kamenev-Type Oscillation Criteria for Higher-Order Neutral Delay Dynamic Equations

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Abstract

In this paper, we are primarily concerned with the oscillation of solutions to higher-order dynamic equations. We first extend some recent results which have been obtained for second-order dynamic equations to even-order neutral delay dynamic equations. Our technique depends on making use of the Riccati substitution and the use of properties of the generalized Taylor monomials on time scales. We also extend our results to odd-order neutral delay dynamic equations. Some examples for differential equations, difference equations, and q-difference equations (which have important applications in quantum theory) are given to illustrate the results obtained.

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1 Introduction

The theory of time scales has received a great deal of attention since it was introduced by Hilger in his Ph.D. Thesis in 1988. The goal is to unify and extend continuous and discrete analysis (see the Ph.D. Thesis of Hilger [9]). Since then, many authors have contributed to various aspects of this new theory; see the survey paper [2] by Agarwal et al. and references cited therein. We refer the readers to the book [4] by Bohner and Peterson, which summarizes and organizes much of the time scale theory.

A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to a number of applications, among them the study of population dynamic models which are discrete in season (and may follow a difference scheme with variable step-size or may be modeled by continuous dynamic systems). Then the population dies out, say in winter, while the eggs are incubating or are dormant, and then in season again, the eggs hatch and therefore this model gives rise to a non-overlapping population (see [4, Example 1.39]). Not only does this new theory of so-called "dynamic equations" unify the existing theories of differential equations and of difference equations, but it also extends these classical cases to cases "in between", e.g., to the so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$ where $q > 1$ (which has important applications in quantum theory (see the book [10] by Kac and Cheung). Moreover, these ideas can be applied to many different types of time scales such as $\mathbb{T} = h\mathbb{Z}$ $(h > 0)$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \Big\{\sum_{k=1}^{n} 1/k : n \in \mathbb{N}\Big\}$, the set of harmonic numbers.

For a reader not familiar with time scale calculus, we summarize some of the following basic information. A *time scale*, which inherits the standard topology on $\mathbb{R}$, is a nonempty closed subset of the real line. Here, and later throughout this paper, a time scale will be denoted by the symbol $\mathbb{T}$, and intervals with a subscript are used to denote the intersection of the usual real interval with $\mathbb{T}$. For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf(t, \infty)_{\mathbb{T}}$ while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup(-\infty, t)_{\mathbb{T}}$ (Here we define $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$). The *graininess function* $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is defined by $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is called *right-dense* if $\sigma(t) = t$ and $t \neq \sup \mathbb{T}$. If $\sigma(t) > t$ it is called *right-scattered*. Similarly *left-dense* and *left-scattered* points are defined with respect to the backward jump operator. We recall also that $\mathbb{T}^\kappa := \mathbb{T}\backslash\{\sup \mathbb{T}\}$ if $\sup \mathbb{T} = \max \mathbb{T}$ and satisfies $\rho(\max \mathbb{T}) \neq \max \mathbb{T}$; otherwise, $\mathbb{T}^\kappa := \mathbb{T}$. The (*Hilger*) *derivative* of a function $f : \mathbb{T} \to \mathbb{R}$ is defined by

$$f^\Delta(t) := \begin{cases} \dfrac{f^\sigma(t) - f(t)}{\mu(t)}, & \mu(t) > 0 \\ \lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t - s}, & \mu(t) = 0 \end{cases}$$

for $t \in \mathbb{T}^\kappa$ (provided that the limit exists). A function f is said to be *rd-continuous* provided that it is continuous at right-dense points in $\mathbb{T}$, and has a finite limit at left-dense points. The *set of rd-continuous functions* will be denoted by $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$. The set of functions $\mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R})$ denotes those functions f whose derivative is in $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$. For $s, t \in \mathbb{T}$ and a function $f \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$, the Δ-integral of f is defined by

$$\int_s^t f(\eta)\Delta\eta = F(t) - F(s) \quad \text{for } s, t \in \mathbb{T},$$

where $F \in \mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R})$ is an anti-derivative of f, i.e., $F^\Delta = f$ on $\mathbb{T}^\kappa$.

In recent years, there have been several papers which have studied oscillation and nonoscillation properties of delay dynamic equations on arbitrary time scales (see the papers [2,3,14]). Much of the work thus far has been motivated by the qualitative theory of second-order equations. There are fewer results dealing with the oscillation and asymptotic behaviour of third and/or higher-order dynamic equations (see the papers [7,12,13]).

In this paper, we shall present some results dealing with the oscillation and asymptotic behaviour of solutions of higher-order neutral delay dynamic equations of the following form:

$$\left[x(t) + A(t)x(\alpha(t))\right]^{\Delta^n} + B(t)x(\beta(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{1.1}$$

Here $n \in \mathbb{N}$, $\mathbb{T}$ is a time scale which satisfies $\sup \mathbb{T} = \infty$, and t_0 is a fixed point in $\mathbb{T}$. In addition, we assume $A \in \mathrm{C_{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $B \in \mathrm{C_{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}_0^+)$. We suppose that $\alpha \in \mathrm{C_{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ and $\beta \in \mathrm{C^1_{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ are strictly increasing and satisfy $\lim_{t\to\infty} \alpha(t) = \infty$, $\lim_{t\to\infty} \beta(t) = \infty$, and $\alpha(t) \le t$, $\beta(t) \le t$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. We also assume that $\beta([t_0, \infty)_{\mathbb{T}}) = [\beta(t_0), \infty)_{\mathbb{T}}$. Our method makes use of the Riccati substitution technique and, after deducing some second-order dynamic inequalities, we relate oscillation of (1.1) to second-order dynamic equations, with which we are familiar from well-known results in the literature.

We set $t_{-1} := \min\left\{\inf_{t\in[t_0,\infty)_{\mathbb{T}}}\{\alpha(t)\}, \inf_{t\in[t_0,\infty)_{\mathbb{T}}}\{\beta(t)\}\right\}$. By a solution of (1.1) we mean a function $x \in \mathrm{C_{rd}}([t_{-1}, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $x + A(t)x \circ \alpha \in \mathrm{C^n_{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and x satisfies (1.1) identically on $[t_0, \infty)_{\mathbb{T}}$. If x is a solution of (1.1), then x is said to be *nonoscillatory* if x is eventually of one sign. Otherwise, x is said to be *oscillatory*.

Let us briefly recall some classical results for the qualitative theory of the second-order differential equation

$$x''(t) + A(t)x(t) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{R}}, \tag{1.2}$$

where $A \in \mathrm{C_{rd}}([t_0, \infty), \mathbb{R}_0^+)$.

In the past two decades, obtaining sufficient conditions for the oscillation and nonoscillation of solutions of second-order differential equations has attracted a great deal of attention, and one of the most powerful tools is the integral averaging technique. These ideas may be traced back to the classical result due to Wintner in 1949 (see [17]), where

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}(t-s)A(s)\mathrm{d}s=\infty$$

is shown to be a sufficient condition for the oscillation of all solutions to (1.2). Later on, in 1952, Hartman (see [8]) showed that the limit condition given above may not be replaced by a $\limsup$ condition. Specifically, it was shown that

$$-\infty<\liminf_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}(t-s)A(s)\mathrm{d}s<\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}(t-s)A(s)\mathrm{d}s\leq\infty$$

implies that every solution of (1.2) is oscillatory.

In 1978, Kamenev (see [11]) improved the result of Wintner by showing that the condition

$$\lim_{t\to\infty}\frac{1}{t^k}\int_{t_0}^{t}(t-s)^kA(s)\mathrm{d}s=\infty\quad\text{for some } k\in\mathbb{N}$$

is sufficient for the oscillation of all solutions to (1.2).

Analogous results have also been obtained for the second-order difference equation

$$\Delta^2x(n)+A(n)x(n)=0\quad\text{for } n\in[n_0,\infty)_{\mathbb{Z}},\tag{1.3}$$

where Δ denotes the forward difference operator and $\{A(n)\}$ is a nonnegative sequence. It was shown that

$$\lim_{n\to\infty}\frac{1}{n^k}\sum_{i=n_0}^{n-1}(n-i)^kA(i)=\infty\quad\text{for some } k\in\mathbb{N}\tag{1.4}$$

is sufficient for the oscillation of all solutions of (1.3) (see the papers [15, 18]). We note also that

$$\lim_{n\to\infty}\frac{1}{n^{\underline{k}}}\sum_{i=n_0}^{n-1}(n-i)^{\underline{k}}A(i)=\infty\quad\text{for some } k\in\mathbb{N},$$

where $t^{\underline{k}}:=t(t-1)\cdots(t-k+1)$ is the falling (factorial) function (see the book [16] by Kelley and Peterson), is equivalent to (1.4). For additional results on integral averaging see Erbe [6] and the references therein. In order to view our results in the time scale setting as analogues of classical results, we shall use generalized polynomials instead of the usual polynomials. The main motivation behind our paper is the paper [5] by Džurina. Our results extend and generalize the results in [5] to arbitrary time scales.

The paper is organized as follows: In Section 2, we give some important results required in the sequel some of which are taken from [4]; in Section 3, we prove some oscillation criteria for (1.1) when the order is even; in Section 4, we present some Kamenev-type oscillation criteria for even-order delay dynamic equations with neutral term; and finally in Section 5, we extend our results to neutral delay dynamic equations of odd order.

2 Auxiliary Results

The generalized Taylor monomials $h_k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $k \in \mathbb{N}$, are defined by

$$h_k(t,s) := \begin{cases} 1, & k = 0 \\ \displaystyle\int_s^t h_{k-1}(\eta, s)\Delta\eta, & k \in \mathbb{N} \end{cases} \tag{2.1}$$

for all $s, t \in \mathbb{T}$. Note that these generalized Taylor monomials satisfy

$$\frac{\partial}{\Delta t} h_k(t,s) = \begin{cases} 0, & k = 0 \\ h_{k-1}(t,s), & k \in \mathbb{N} \end{cases} \tag{2.2}$$

for all $s, t \in \mathbb{T}$ and $k \in \mathbb{N}_0$. Any finite linear combination of generalized Taylor monomials is called a generalized polynomial.

Since the generalized Taylor monomials play a major role in this paper, we next give some of their important properties.

Property 2.1 ([13, Property 1]). Using induction and the definition given by (2.1), it is easy to see that $h_k(t,s) \geq 0$ holds for all $k \in \mathbb{N}_0$ and $s, t \in \mathbb{T}$ with $t \geq s$ and $(-1)^k h_k(t,s) \geq 0$ holds for all $k \in \mathbb{N}$ and $s, t \in \mathbb{T}$ with $t \leq s$. In view of the fact (2.2), for all $k \in \mathbb{N}$, it is evident that $h_k(t,s)$ is increasing in t provided that $t \geq s$, and $(-1)^k h_k(t,s)$ is decreasing in t provided that $t \leq s$. In addition, for all $s, t \in \mathbb{T}$ and all $k, l \in \mathbb{N}_0$ with $l \leq k$, $h_k(t,s) \leq (t-s)^{k-l} h_l(t,s)$ holds when $t \geq s$, while $(-1)^k h_k(t,s) \leq (-1)^l (s-t)^{k-l} h_l(t,s)$ when $t \leq s$.

Using L'Hôpital's rule (see [4, Theorem 1.120]), we have the following corollary.

Remark 2.1. Let $\sup \mathbb{T} = \infty$. Then for any $k \in \mathbb{N}$ and fixed $r, s \in \mathbb{T}$, we have

$$\lim_{t \to \infty} \frac{h_k(t,r)}{h_k(t,s)} = 1.$$

With the following lemma, we are able to give an alternative equivalent formulation of the generalized Taylor monomials.

Lemma 2.2. *The generalized Taylor monomials* $h_k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ *satisfy*

$$h_k(t,s) = \begin{cases} 1, & k = 0 \\ \displaystyle\int_s^t h_{k-1}(t, \sigma(\eta))\Delta\eta, & k \in \mathbb{N} \end{cases} \tag{2.3}$$

for all $s, t \in \mathbb{T}$.

Proof. By Taylor's formula (see [4, Lemma 1.109, Theorems 1.111–1.113]), for any $k \in \mathbb{N}_1$ and fixed $r, s \in \mathbb{T}$, we have

$$\begin{aligned} h_k(t,s) &= \sum_{l=0}^{k-1} h_l(t,r) h_k^{\Delta_t^l}(r,s) + \int_r^t h_{k-1}(t,\sigma(\eta)) h_k^{\Delta_t^k}(\eta,s)\Delta\eta \\ &= \sum_{l=0}^{k-1} h_l(t,r) h_{k-l}(r,s) + \int_r^t h_{k-1}(t,\sigma(\eta))\Delta\eta \end{aligned}$$

for all $t \in \mathbb{T}$, which gives (2.3) by letting $r = s$. □

Corollary 2.3. *For all $s, t \in \mathbb{T}$ and $k \in \mathbb{N}$, we have*

$$h_k^{\Delta_s}(t,s) = -h_{k-1}(t,\sigma(s)).$$

In the sequel, we use the following two important lemmas.

Lemma 2.4 (Change of order of integration [12, Lemma 1])**.** *Assume that $s, t \in \mathbb{T}$ and $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}\times\mathbb{T},\mathbb{R})$, the*

$$\int_s^t \int_\eta^t f(\eta,\xi)\Delta\xi\Delta\eta = \int_s^t \int_s^{\sigma(\xi)} f(\eta,\xi)\Delta\eta\Delta\xi.$$

Lemma 2.5 (Substitution [4, Theorem 1.93])**.** *Assume that $f \in \mathrm{C}_{\mathrm{rd}}^1(\widetilde{\mathbb{T}},\mathbb{R})$ and $g \in \mathrm{C}_{\mathrm{rd}}^1(\mathbb{T},\mathbb{R})$ such that $g^\Delta > 0$ on $\mathbb{T}$ and let $\widetilde{\mathbb{T}} := g(\mathbb{T})$ be a time scale. Then, $(f\circ g)^\Delta = (f^{\widetilde{\Delta}}\circ g)g^\Delta$ holds on $\mathbb{T}$.*

The following remark can be extracted from the proof of the one above.

Remark 2.6. Assume that $g \in \mathrm{C}_{\mathrm{rd}}^1(\mathbb{T},\mathbb{R})$ satisfies $g^\Delta > 0$ on $\mathbb{T}^\kappa$, and that $g([s,t]_\mathbb{T}) = [g(s),g(t)]_\mathbb{T}$ holds for some fixed $s,t \in \mathbb{T}$ with $t > s$. Then for $f \in \mathrm{C}_{\mathrm{rd}}^1([g(s),g(t)]_\mathbb{T}, \mathbb{R})$, we have $(f\circ g)^\Delta = (f^\Delta \circ g)g^\Delta$ on $[s,t]_{\mathbb{T}^\kappa}$.

In general the chain rule for the time scale calculus is not the same as in the ordinary calculus case. Remark 2.6 gives us conditions under which the chain rule for time scale calculus coincides with the usual calculus on the real line.

One of the most powerful tools for higher-order dynamic equations (particularly for difference and differential equations) is the following one, which is known as Kiguradze's theorem.

Theorem 2.7 (Kiguradze's Theorem [1, Theorem 5])**.** *Let $\sup\mathbb{T} = \infty$, $n \in \mathbb{N}$ and $f \in \mathrm{C}_{\mathrm{rd}}^{\mathrm{n}}(\mathbb{T},\mathbb{R}^+)$. Suppose that $f^{\Delta^n} \not\equiv 0$ is either nonnegative or nonpositive on $\mathbb{T}$. Then there exists $m \in [0,n)_\mathbb{Z}$ such that $(-1)^{n-m} f^{\Delta^n}(t) \ge 0$ holds for all sufficiently large t. Moreover, both of the following conditions hold:*

(i) $0 \le k < m$ *implies* $f^{\Delta^k}(t) > 0$ *for all sufficiently large* t,

(ii) $m \leq k < n$ *implies* $(-1)^{m-k} f^{\Delta^k}(t) > 0$ *for all sufficiently large* t.

In what follows, we prove the following lemma, which plays a crucial role in the sequel.

Lemma 2.8. *Let* $\sup \mathbb{T} = \infty$ *and* $f \in \mathrm{C}^{\mathrm{n}}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^+)$ $(n \geq 2)$*. Moreover, suppose that Kiguradze's theorem holds with* $m \in [1, n)_{\mathbb{N}}$ *and* $f^{\Delta^n} \leq 0$ *on* $\mathbb{T}$*. Then, there exists a sufficiently large* $t_1 \in \mathbb{T}$ *such that*

$$f^{\Delta}(t) \geq h_{m-1}(t, t_1) f^{\Delta^m}(t) \quad \textit{for all } t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.4}$$

Proof. Note that f^{Δ^m} is nonincreasing on $\mathbb{T}$. We shall give the proof for the case when $m \in [2, n)_{\mathbb{N}}$, since (2.4) holds trivially by the definition of generalized polynomials for $m = 1$. Using the Taylor formula for f^{Δ} centered at $t_1 \in \mathbb{T}$, Property 2.1 and the eventually decreasing nature of f^{Δ^m} on $\mathbb{T}$, we have

$$\begin{aligned} f^{\Delta}(t) &= \sum_{k=1}^{m-2} h_k(t, t_1) f^{\Delta^k}(t_1) + \int_{t_1}^{t} h_{m-2}(t, \sigma(\eta)) f^{\Delta^m}(\eta) \Delta\eta \\ &\geq \left(\int_{t_1}^{t} h_{m-2}(t, \sigma(\eta)) \Delta\eta \right) f^{\Delta^m}(t) \end{aligned}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$ (see [4, Lemma 1.109, Theorems 1.111–1.113]). An application of Lemma 2.2 gives (2.4), and completes the proof. $\square$

The following important result may be found in [1].

Lemma 2.9 ([1, Lemma 7]). *Let* $\sup \mathbb{T} = \infty$, $n \in \mathbb{N}$ *and* $f \in \mathrm{C}^{\mathrm{n}}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$*. Then the following assertions hold:*

(i) $\liminf_{t \to \infty} f^{\Delta^n}(t) > 0$ *implies* $\lim_{t \to \infty} f^{\Delta^k}(t) = \infty$ *for all* $k \in [0, n)_{\mathbb{N}_0}$.

(ii) $\limsup_{t \to \infty} f^{\Delta^n}(t) < 0$ *implies* $\lim_{t \to \infty} f^{\Delta^k}(t) = -\infty$ *for all* $k \in [0, n)_{\mathbb{N}_0}$.

Due to Kiguradze's theorem and Lemma 2.9, we infer the following corollary.

Corollary 2.10. *Let* f *be as in Kiguradze's theorem. Then we have*

$$\lim_{t \to \infty} f^{\Delta^k}(t) = 0 \quad \textit{for all } k \in (m, n)_{\mathbb{N}}.$$

3 Oscillation Criteria for Even-Order Equations

In this section, we give some oscillation criteria for (1.1) for the even-order case.

We first introduce

$$B_k^{(1)}(t) := \begin{cases} B(t)\big[1 - A(\alpha(t))\big], & k = n - 1 \\ (-1)^{n-k-2} \displaystyle\int_t^{\infty} h_{n-k-2}(t, \sigma(\eta)) B_{n-1}^{(1)}(\eta) \Delta\eta, & \text{otherwise} \end{cases} \tag{3.1}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$. Here $\cdot^{(1)}$ represents an upper index. Our first main result reads as follows.

Theorem 3.1. *Assume that* $n \in 2\mathbb{N}$ *and* $A \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, [0, 1]_{\mathbb{R}})$*, and that for every* $k \in (0, n)_{2\mathbb{N}-1}$*, there exists a function* $\varphi_k \in \mathrm{C}^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+_0)$ *such that*

$$\limsup_{t\to\infty} \int_r^t \left(B_k^{(1)}(\eta)\varphi_k(\eta) - \frac{\left(\varphi_k^{\Delta}(\eta)\right)^2}{4\varphi_k(\eta)\beta^{\Delta}(\eta)h_{k-1}(\beta(\eta), s)} \right)\Delta\eta = \infty \tag{3.2}$$

for every sufficiently large but fixed $r, s \in [t_0, \infty)_{\mathbb{T}}$ *with* $\beta(r) > s$*. Then every solution of* (1.1) *is oscillatory.*

Proof. Without loss of generality suppose that x is an eventually positive solution of (1.1). Then we may pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t), x(\alpha(t)), x(\beta(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Set

$$y_x(t) := x(t) + A(t)x(\alpha(t)) \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.3}$$

Then, $y_x(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. From (1.1), we have

$$y_x^{\Delta^n}(t) = -B(t)x(\beta(t)) \leq 0 \quad \text{for all } t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.4}$$

It follows from Kiguradze's theorem that there exist a sufficiently large $t_2 \in [t_1, \infty)_{\mathbb{T}}$ and an odd integer $m \in [1, n)_{\mathbb{N}_0}$ such that for all $t \in [t_2, \infty)_{\mathbb{T}}$, $y_x^{\Delta^k}(t) > 0$ for all $k \in [0, m)_{\mathbb{N}_0}$ and $(-1)^{m-k}y_x^{\Delta^k}(t) > 0$ for all $k \in [m, n)_{\mathbb{N}}$. In particular, we have $y_x(t) \geq x(t) > 0$ and $y_x^{\Delta}(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Because of $\alpha \circ \beta \leq \beta$ on $[t_2, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} x(\beta(t)) =& y_x(\beta(t)) - A(\beta(t))x(\alpha(\beta(t))) \\ \geq& y_x(\beta(t)) - A(\beta(t))y_x(\alpha(\beta(t))) \\ \geq& \big[1 - A(\beta(t))\big]y_x(\beta(t)) \end{aligned} \tag{3.5}$$

for all $t \in [t_2, \infty)_{\mathbb{T}}$. Using Corollary 2.10, we get from integrating (3.4) over $[t, \infty)_{\mathbb{T}} \subset [t_2, \infty)_{\mathbb{T}}$ for a total of $(n - m - 1)$ times and changing the order of integration from the innermost integral to the outermost one, we have

$$\begin{aligned} (-1)^{n-m-1}y_x^{\Delta^{m+1}}(t) =& -\int_t^{\infty}\int_{\eta_{n-m-2}}^{\infty}\cdots\int_{\eta_2}^{\infty} B(\eta_1)x(\beta(\eta_1))\Delta\eta_1\cdots\Delta\eta_{n-2}\Delta\eta_{n-m-1} \\ =& (-1)^{n-m-1}\int_t^{\infty} h_{n-m-2}(t, \sigma(\eta))B(\eta)x(\beta(\eta))\Delta\eta \end{aligned} \tag{3.6}$$

for all $t \in [t_2, \infty)_{\mathbb{T}}$. Substituting (3.5) into (3.6), and using the increasing nature of $y_x \circ \beta$ (both y_x and β are increasing) and (3.1), we obtain

$$y_x^{\Delta^{m+1}}(t) + B_m^{(1)}(t)y_x(\beta(t)) \leq 0 \quad \text{for all } t \in [t_2, \infty)_{\mathbb{T}}. \tag{3.7}$$

We notice that the term $(-1)^{n-m-1}$ disappears since $(n-m-1)$ is even. Now, define

$$Y_m(t) := \frac{\varphi_m(t) y_x^{\Delta^m}(t)}{y_x(\beta(t))} > 0 \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}. \tag{3.8}$$

Using (3.8), the positivity of φ_m, the increasing nature of $y_x \circ \beta$ and the decreasing nature of $y_x^{\Delta^m}$, we obtain

$$\frac{\varphi_m(t) y_x^{\Delta^m}(\beta(t))}{y_x(\beta(t))} \geq Y_m(t) \geq \frac{\varphi_m(t) y_x^{\Delta^m \sigma}(t)}{y_x(\beta(\sigma(t)))} = \frac{\varphi_m(t) Y_m^{\sigma}(t)}{\varphi_m^{\sigma}(t)} \quad \text{for all } t \in [t_2, \infty)_{\mathbb{T}}. \tag{3.9}$$

Hence, from Remark 2.6, (3.7) and (3.8), we get

$$\begin{aligned} Y_m^{\Delta}(t) =& \frac{\varphi_m(t) y_x^{\Delta^{m+1}}(t)}{y_x(\beta(t))} + \frac{y_x^{\Delta^m \sigma}(t)\Big[\varphi_m^{\Delta}(t) y_x(\beta(t)) - \varphi_m(t)\big(y_x(\beta(t))\big)^{\Delta}\Big]}{y_x(\beta(t)) y_x(\beta(\sigma(t)))} \\ =& - B_m^{(1)}(t)\varphi_m(t) + \left(\varphi_m^{\Delta}(t) - \frac{\varphi_m(t)\beta^{\Delta}(t) y_x^{\Delta}(\beta(t))}{y_x(\beta(t))}\right)\frac{Y_m^{\sigma}(t)}{\varphi_m^{\sigma}(t)} \end{aligned} \tag{3.10}$$

for all $t \in [t_2, \infty)_{\mathbb{T}}$. By Lemma 2.8, for some $t_3 \in [t_2, \infty)_{\mathbb{T}}$, we have

$$y_x^{\Delta}(t) \geq h_{m-1}(t, t_3) y_x^{\Delta^m}(t) \tag{3.11}$$

for all $t \in [t_3, \infty)_{\mathbb{T}}$. Using (3.9), (3.10) and (3.11), for all $t \in [t_4, \infty)_{\mathbb{T}}$, where $\beta(t_4) > t_3$, we get

$$\begin{aligned} Y_m^{\Delta}(t) \leq& - B_m^{(1)}(t)\varphi_m(t) + \left(\varphi_m^{\Delta}(t) - \varphi_m(t)\beta^{\Delta}(t) h_{m-1}(\beta(t), t_3)\frac{y_x^{\Delta^m}(\beta(t))}{y_x(\beta(t))}\right)\frac{Y_m^{\sigma}(t)}{\varphi_m^{\sigma}(t)} \\ \leq& - B_m^{(1)}(t)\varphi_m(t) + \left(\varphi_m^{\Delta}(t) - \varphi_m(t)\beta^{\Delta}(t) h_{m-1}(\beta(t), t_3)\frac{Y_m^{\sigma}(t)}{\varphi_m^{\sigma}(t)}\right)\frac{Y_m^{\sigma}(t)}{\varphi_m^{\sigma}(t)} \\ \leq& - B_m^{(1)}(t)\varphi_m(t) + \frac{\big(\varphi_m^{\Delta}(t)\big)^2}{4\varphi_m(t)\beta^{\Delta}(t) h_{m-1}(\beta(t), t_3)}, \end{aligned} \tag{3.12}$$

which yields by integrating over $[t_4, \infty)_{\mathbb{T}}$ that

$$\int_{t_4}^{\infty} \left(B_m^{(1)}(\eta)\varphi_m(\eta) - \frac{\big(\varphi_m^{\Delta}(\eta)\big)^2}{4\varphi_m(\eta)\beta^{\Delta}(\eta) h_{m-1}(\beta(\eta), t_3)}\right)\Delta\eta \leq Y_m(t_4).$$

This contradicts (3.2), the proof is hence completed. □

The next result follows from the previous one.

Corollary 3.2. *Assume that* $n \in 2\mathbb{N}$ *and* $A \in \mathrm{C_{rd}}([t_0, \infty)_{\mathbb{T}}, [0, 1]_{\mathbb{R}})$*, and*

$$\limsup_{t \to \infty} \int_r^t \left(B_k^{(1)}(\eta) h_k(\eta, s) - \frac{h_{k-1}(\eta, s)}{4h_k(\beta(\eta), s)}\right)\Delta\eta = \infty \tag{3.13}$$

for every $k \in (0, n)_{2\mathbb{N}-1}$ *and every sufficiently large but fixed* $r, s \in [t_0, \infty)_{\mathbb{T}}$ *with* $\beta(r) > s$*. Then, every solution of* (1.1) *is oscillatory.*

Next, we have an illustrative example for the difference equation case.

Example 3.3. Consider the following even-order difference equation:

$$\Delta^n\big[x(t)+a_0x(t-\alpha_0)\big]+\frac{b_0}{h_n(t,1-n)}x(t-\beta_0)=0 \quad \text{for } t\in[1,\infty)_{\mathbb{Z}}, \tag{3.14}$$

where $a_0\in[0,1)_{\mathbb{R}}$, $\alpha_0\in\mathbb{N}_0$, $b_0\in\mathbb{R}^+$ and $\beta_0\in\mathbb{N}_0$. For $s,t\in[1,\infty)_{\mathbb{Z}}$ and $k\in\mathbb{N}$, we have

$$h_k(t,s)=\frac{(t-s)^{\underline{k}}}{k!}.$$

It is easy to show that

$$\Delta\bigg(\frac{1}{h_k(t,s)}\bigg)=-\frac{k}{k+1}\frac{1}{h_{k+1}(t,s-1)}$$

holds, and this implies

$$\sum_{\eta=t}^{\infty}\frac{1}{h_k(\eta,s)}=\frac{k}{k-1}\frac{1}{h_{k-1}(t,s+1)}$$

for all $k\in[2,\infty)_{\mathbb{N}}$. Thus, for $t\in[1,\infty)_{\mathbb{Z}}$, we can compute for $k\in(0,n)_{2\mathbb{N}-1}$ that

$$B_k^{(1)}(t)=\frac{\lambda(n,k)b_0(1-a_0)}{h_{k+1}(t,-k)}, \quad \text{where} \quad \lambda(n,k):=\frac{n}{k+1},$$

which yields

$$\sum_{\eta=1}^{t-1}\bigg(B_k^{(1)}(\eta)h_k(\eta,-k)-\frac{h_{k-1}(\eta,-k)}{4h_k(\eta-\beta_0,-k)}\bigg)\approx\bigg(nb_0(1-a_0)-\frac{k}{4}\bigg)\sum_{\eta=1}^{t-1}\frac{1}{\eta}.$$

By Corollary 3.2 and Remark 2.1, every solution of (3.14) oscillates, if

$$\min_{k\in\{1,3,\ldots,n-1\}}\bigg\{nb_0(1-a_0)-\frac{k}{4}\bigg\}=nb_0-\frac{n-1}{4}>0\Rightarrow b_0(1-a_0)>\frac{n-1}{4n}$$

since (3.13) holds for every $k\in(0,n)_{2\mathbb{N}-1}$ and every sufficiently large $s\in[1,\infty)_{\mathbb{Z}}$.

For the following theorem, we need to introduce

$$B_k^{(2)}(t):=\begin{cases}B(t), & k=n-1\\ (-1)^{n-k-2}\displaystyle\int_t^{\infty}h_{n-k-2}(t,\sigma(\eta))B_{n-1}^{(2)}(\eta)\Delta\eta, & \text{otherwise}\end{cases} \tag{3.15}$$

for $t\in[t_0,\infty)_{\mathbb{T}}$. Here $\cdot^{(2)}$ is an upper index.

Theorem 3.4. *Assume that* $n \in 2\mathbb{N}$ *and* $A \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[-1,0]_{\mathbb{R}})$ *with* $\liminf_{t\to\infty} A(t) > -1$*, and that for every* $k \in (0,n)_{2\mathbb{N}-1}$*, there exists a function* $\varphi_k \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+_0)$ *such that*

$$\limsup_{t\to\infty}\int_r^t\left(B^{(2)}_k(\eta)\varphi_k(\eta)-\frac{\big(\varphi^{\Delta}_k(\eta)\big)^2}{4\varphi_k(\eta)\beta^{\Delta}(\eta)h_{k-1}(\beta(\eta),s)}\right)\Delta\eta=\infty \tag{3.16}$$

for every sufficiently large but fixed $r,s \in [t_0,\infty)_{\mathbb{T}}$ *with* $\beta(r) > s$*. Then every solution of* (1.1) *oscillates or tends to zero asymptotically.*

Proof. Without loss of generality suppose that x is an eventually positive solution of (1.1) which does not tend to zero asymptotically. Let $t_1 \in [t_0,\infty)_{\mathbb{T}}$ and $a_0 \in [0,1)_{\mathbb{R}}$ satisfy $x(t), x(\alpha(t)), x(\beta(t)) > 0$ and $A(t) \geq -a_0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. Let y_x be defined on $[t_1,\infty)_{\mathbb{T}}$ by (3.3). From (1.1), we have (3.4) on $[t_1,\infty)_{\mathbb{T}}$. Then, for each $k \in [0,n)_{\mathbb{N}_0}$, $y_x^{\Delta^k}$ is of fixed sign on $[t_2,\infty)_{\mathbb{T}}$ for some $t_2 \in [t_1,\infty)_{\mathbb{T}}$. Hence, $\ell_y := \lim\limits_{t\to\infty} y_x(t)$ exists. Below, we shall show that $\ell_y > 0$. First consider the case where x is unbounded. Then we may pick an increasing divergent sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset [t_2,\infty)_{\mathbb{T}}$ such that $x(\xi_k) = \max_{t\in[t_0,\xi_k]_{\mathbb{T}}}\{x(t)\}$ for all $k \in \mathbb{N}$ and $\{x(\xi_k)\}_{k\in\mathbb{N}}$ is increasing and divergent. Then, we have $y_x(\xi_k) \geq (1-a_0)x(\xi_k)$ for all $k \in \mathbb{N}$, which yields $\ell = \infty$ by letting $k \to \infty$. Next consider the case where x is bounded. Then we may pick an increasing divergent sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset [t_2,\infty)_{\mathbb{T}}$ such that $\lim_{k\to\infty} x(\xi_k) = \ell_x$, where $\ell_x := \limsup_{t\to\infty} x(t)$. It is clear that $\limsup_{t\to\infty} x(\alpha(t)) \leq \ell_x$, and $\ell_x > 0$ since x does not tend to zero asymptotically. Then, we have $y_x(\xi_k) \geq x(\xi_k) - a_0x(\alpha(\xi_k))$ for all $k \in \mathbb{N}$, which yields $\ell_y \geq (1-a_0)\ell_x > 0$ by letting $k \to \infty$. In both cases, we obtain $y_x > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Then it follows from Kiguradze's theorem that there exists an odd integer $m \in [1,n)_{\mathbb{N}_0}$ such that for all $t \in [t_2,\infty)_{\mathbb{T}}$, $y_x^{\Delta^k}(t) > 0$ for all $k \in [0,m)_{\mathbb{N}_0}$ and $(-1)^{m-k}y_x^{\Delta^k}(t) > 0$ for all $k \in [m,n)_{\mathbb{N}}$. In particular, we have $x \geq y_x > 0$ and $y_x^{\Delta} > 0$ on $[t_2,\infty)_{\mathbb{T}}$. The remainder of the proof is very similar to that in the proof of Theorem 3.1. So we obtain

$$\int_{t_4}^{\infty}\left(B^{(2)}_m(\eta)\varphi_m(\eta)-\frac{\big(\varphi^{\Delta}_m(\eta)\big)^2}{4\varphi_m(\eta)\beta^{\Delta}(\eta)h_{m-1}(\beta(\eta),t_3)}\right)\Delta\eta\leq Y_m(t_4),$$

where Y_m is defined on $[t_2,\infty)_{\mathbb{T}}$ by (3.8), for some fixed $t_4 \in [t_3,\infty)_{\mathbb{T}}$ with $\beta(t_4) > t_3$ and some fixed sufficiently large $t_3 \in [t_2,\infty)_{\mathbb{T}}$. This contradicts (3.16) and completes the proof. □

The result below follows from the previous theorem.

Corollary 3.5. *Assume that* $A \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[-1,0]_{\mathbb{R}})$ *with* $\liminf_{t\to\infty} A(t) > -1$*, and*

$$\limsup_{t\to\infty}\int_r^t\left(B^{(2)}_k(\eta)h_k(\eta,s)-\frac{h_{k-1}(\eta,s)}{4h_k(\beta(\eta),s)}\right)\Delta\eta=\infty \tag{3.17}$$

for every $k \in (0,n)_{2\mathbb{N}-1}$ *and every sufficiently large but fixed* $r,s \in [t_0,\infty)_{\mathbb{T}}$ *with* $\beta(r) > s$*. Then, every solution of* (1.1) *oscillates or tends to zero asymptotically.*

Another illustrative example for Theorem 3.4 is given below on the quantum set.

Example 3.6. Assume $q > 1$ and consider the following even-order q-difference equation:

$$\mathrm{D}_q^n\big[x(t) + a_0 x(q^{-\alpha_0}t)\big] + \frac{b_0}{h_n(t, q^{1-n})} x(q^{-\beta_0}t) = 0 \quad \text{for } t \in [1, \infty)_{q^{\mathbb{Z}} \cup \{0\}}, \tag{3.18}$$

where $a_0 \in (-1, 0]_{\mathbb{R}}$, $\alpha_0 \in \mathbb{N}_0$, $b_0 \in \mathbb{R}^+$ and $\beta_0 \in \mathbb{N}_0$. Here, for all $s, t \in [1, \infty)_{q^{\mathbb{Z}} \cup \{0\}}$ and all $k \in \mathbb{N}_0$, we have

$$\mathrm{D}_q^k x(t) := \begin{cases} \dfrac{x(qt) - x(t)}{(q-1)t}, & k = 1 \\ \mathrm{D}_q \mathrm{D}_q^{k-1} x(t), & k \in [2, \infty)_{\mathbb{N}} \end{cases} \quad \text{and} \quad h_k(t, s) := \prod_{j=0}^{k-1} \frac{t - q^j s}{\sum_{i=0}^{j} q^i}.$$

For $s, t \in [1, \infty)_{q^{\mathbb{Z}} \cup \{0\}}$ and $k \in \mathbb{N}$, it is not difficult to show that

$$\mathrm{D}_q\left(\frac{1}{h_k(t, qs)}\right) = \frac{q^k - 1}{q^k (q^{k+1} - 1) h_{k+1}(t, s)},$$

which implies

$$\int_t^\infty \frac{1}{h_k(\eta, q^{-1}s)} \mathrm{d}_q\eta = \frac{q^{k-1}(q^k - 1)}{(q^{k-1} - 1) h_{k-1}(t, s)}$$

for all $k \in [2, \infty)_{\mathbb{N}}$. Therefore, for $t \in [1, \infty)_{q^{\mathbb{Z}} \cup \{0\}}$ and $k \in (0, n)_{2\mathbb{N}-1}$, we have

$$B_k^{(2)}(t) = \frac{\lambda(n, k) b_0}{h_{k+1}(t, q^{-k})}, \quad \text{where} \quad \lambda(n, k) := \frac{q^{\frac{n(n-1)}{2}}(q^n - 1)}{q^{\frac{(k+1)k}{2}}(q^{k+1} - 1)}.$$

For $t \in [1, \infty)_{q^{\mathbb{Z}} \cup \{0\}}$, we have

$$\begin{aligned} &\int_1^t \left(B_k^{(2)}(\eta) h_k(\eta, q^{-k}) - \frac{h_{k-1}(\eta, q^{-k})}{4 h_k(q^{-\beta_0}\eta, q^{-k})}\right) \mathrm{d}_q\eta \\ =&(q-1) \int_1^t \left(\frac{q^{\frac{n(n-1)}{2}}(q^{n-1} - 1) b_0}{q^{\frac{(k+1)k}{2}}(q^{k+1} - 1)^2 (\eta - 1)} - \frac{1}{4(q^k - 1)(q^{-\beta_0}\eta - q^{-1})}\right) \mathrm{d}_q\eta \\ \approx&(q-1) \left(\frac{q^{\frac{n(n-1)}{2}}(q^{n-1} - 1) b_0}{q^{\frac{(k+1)k}{2}}(q^{k+1} - 1)^2} - \frac{q^{\beta_0}}{4(q^k - 1)}\right) \int_1^t \frac{1}{\eta} \mathrm{d}_q\eta. \end{aligned}$$

for all $k \in (0, n)_{2\mathbb{N}-1}$. Due to Corollary 3.2 and Remark 2.1, every solution of (3.18) is oscillatory provided that

$$\min_{k \in \{1, 3, \ldots, n-1\}} \left\{\frac{q^{\frac{n(n-1)}{2}}(q^{n-1} - 1) b_0}{q^{\frac{(k+1)k}{2}}(q^{k+1} - 1)^2} - \frac{q^{\beta_0}}{4(q^k - 1)}\right\} = \frac{(q^{n-1} - 1) b_0}{(q^n - 1)^2} - \frac{q^{\beta_0}}{4(q^{n-1} - 1)} > 0$$

or equivalently

$$\frac{b_0}{q^{\beta_0}} > \frac{(q^n - 1)^2}{4(q^{n-1} - 1)^2}.$$

In this present case, one can see that (3.13) holds for every $k \in (0, n)_{2\mathbb{N}-1}$ and every sufficiently large $s \in [1, \infty)_{q^{\mathbb{Z}} \cup \{0\}}$.

4 Kamenev-Type Oscillation Criteria for Even-Order Equations

In this section, we give Kamenev type oscillation criteria for (1.1) for the even-order case.

Theorem 4.1. *Assume that $n \in 2\mathbb{N}$ and $A \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[0,1]_{\mathbb{R}})$. Assume also that for every $k \in (0,n)_{2\mathbb{N}-1}$, there exist a fixed $n_k \in \mathbb{N}$ and a function $\varphi_k \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+_0)$ such that*

$$\limsup_{t\to\infty}\frac{1}{h_{n_k}(t,r)}\int_r^t h_{n_k}(t,\sigma(\eta))\left(B_k^{(1)}(\eta)\varphi_k(\eta)-\frac{\left(\varphi_k^{\Delta}(\eta)\right)^2}{4\varphi_k(\eta)h_{k-1}(\beta(\eta),s)}\right)\Delta\eta=\infty \tag{4.1}$$

for all sufficiently large $r,s \in [t_0,\infty)_{\mathbb{T}}$ with $\beta(r)>s$. Then, every solution of (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 3.1, for all $t \in [t_1,\infty)_{\mathbb{T}} \subset [t_0,\infty)_{\mathbb{T}}$, we get (3.10), where x is positive on $[t_1,\infty)_{\mathbb{T}}$ and Y_m is defined by (3.8) on $[t_1,\infty)_{\mathbb{T}}$. Considering Property 2.1, Corollary 2.3, and integrating by parts, we get

$$\begin{aligned}\int_{t_2}^t Y_m^{\Delta}(\eta)h_{n_m}(t,\sigma(\eta))\Delta\eta =&\left(h_{n_m}(t,\eta)Y_m(\eta)\right)_{\eta=t_2}^{\eta=t}-\int_{t_2}^t Y_m(\eta)h_{n_m}^{\Delta_\eta}(t,\eta)\Delta\eta\\ =&-h_{n_m}(t,t_2)Y_m(t_2)+\int_{t_2}^t Y_m(\eta)h_{n_m-1}(t,\sigma(\eta))\Delta\eta\\ \geq&-h_{n_m}(t,t_2)Y_m(t_2)\end{aligned}$$

for all $t \in [t_2,\infty)_{\mathbb{T}}$, where $\beta(t_2)>t_1$. Using this, multiplying both sides of (3.12) with t replaced by η by $h_{n_m}(t,\sigma(\eta)$, and then integrating as η goes from t_2 to t, we get

$$\begin{aligned}\int_{t_2}^t h_{n_m}(t,\sigma(\eta))\Bigg(&B_m^{(1)}(\eta)\varphi_m(\eta)\\ &-\frac{\left(\varphi_m^{\Delta}(\eta)\right)^2}{4\varphi_m(\eta)\beta^{\Delta}(\eta)h_{m-1}(\beta(\eta),t_1)}\Bigg)\Delta\eta\leq h_{n_m}(t,t_2)Y_m(t_2)\end{aligned}$$

or equivalently, in view of Property 2.1, we have

$$\begin{aligned}\frac{1}{h_{n_m}(t,t_2)}\int_{t_2}^t h_{n_m}(t,\sigma(\eta))\Bigg(&B_m^{(1)}(\eta)\varphi_m(\eta)\\ &-\frac{\left(\varphi_m^{\Delta}(\eta)\right)^2}{4\varphi_m(\eta)\beta^{\Delta}(\eta)h_{m-1}(\beta(\eta),t_1)}\Bigg)\Delta\eta\leq Y_m(t_2),\end{aligned}$$

which contradicts (4.1). The proof is completed. □

As a particular simple result, we have:

Corollary 4.2. *Assume that* $n \in 2\mathbb{N}$ *and* $A \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[0,1]_{\mathbb{R}})$. *And assume for every* $k \in (0,n)_{2\mathbb{N}-1}$ *that*

$$\limsup_{t\to\infty}\frac{1}{h_{n_k}(t,r)}\int_r^t h_{n_k}(t,\sigma(\eta))\bigg(B_k^{(1)}(\eta)h_k(\eta,s) - \frac{h_{k-1}(\eta,s)}{4\varphi_m(\eta)\beta^{\Delta}(\eta)h_{k-1}(\beta(\eta),s)}\bigg)\Delta\eta=\infty \tag{4.2}$$

for some $n_k \in \mathbb{N}$ *and all sufficiently large but fixed* $r,s\in[t_0,\infty)_{\mathbb{T}}$ *with* $\beta(r)>s$. *Then, every solution of* (1.1) *is oscillatory.*

Theorem 4.3. *Assume that* $n \in 2\mathbb{N}$ *and* $A \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[-1,0]_{\mathbb{R}})$ *with* $\liminf_{t\to\infty} A(t) > -1$. *Assume also that for every* $k \in (0,n)_{2\mathbb{N}-1}$, *there exist a fixed* $n_k\in\mathbb{N}$ *and a function* $\varphi_k \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$ *such that*

$$\limsup_{t\to\infty}\frac{1}{h_{n_k}(t,r)}\int_r^t h_{n_k}(t,\sigma(\eta))\bigg(B_k^{(2)}(\eta)\varphi_k(\eta)-\frac{\big(\varphi_k^{\Delta}(\eta)\big)^2}{4\varphi_k(\eta)h_{k-1}(\beta(\eta),s)}\bigg)\Delta\eta=\infty \tag{4.3}$$

for all sufficiently large $r,s\in[t_0,\infty)_{\mathbb{T}}$ *with* $\beta(r)>s$. *Then, every solution of* (1.1) *oscillates or tends to zero asymptotically.*

Now we give the following application of Theorem 4.3 to the case $\mathbb{T}=\mathbb{R}$.

Example 4.4. Consider the following even-order differential equation:

$$\big[x(t)+a_0x(\alpha_0t)\big]^{(n)}+\frac{b_0}{h_n(t,0)}x(\beta_0t)=0 \quad \text{for } t\in[1,\infty)_{\mathbb{R}}, \tag{4.4}$$

where $\cdot^{(n)}$ denotes the usual n-th order derivative, $a_0\in(-1,0]_{\mathbb{R}}$, $\alpha_0\in(0,1]_{\mathbb{R}}$, $b_0\in\mathbb{R}^+$ and $\beta_0\in(0,1]_{\mathbb{R}}$. For $s,t\in[1,\infty)_{\mathbb{R}}$ and $k\in(0,n)_{2\mathbb{N}-1}$. We have

$$h_k(t,s)=\frac{(t-s)^k}{k!}\quad\text{and}\quad B_k^{(2)}(t)=\frac{\lambda(n,k)b_0}{h_{k+1}(t,0)},\quad\text{where}\quad\lambda(n,k):=\frac{n}{k+1},$$

and

$$\begin{aligned}&\frac{1}{h_{n_k}(t,1)}\int_1^t h_{n_k}(t,\sigma(\eta))\bigg(B_k^{(2)}(\eta)h_k(\eta,0)-\frac{h_{k-1}(\eta,0)}{4\beta h_k(\beta_0\eta,0)}\bigg)\mathrm{d}\eta\\ &\approx\bigg(nb_0-\frac{k}{4\beta^{k+1}}\bigg)\frac{1}{(t-1)^{n_k}}\int_1^t\frac{(t-\eta)^{n_k}}{\eta}\mathrm{d}\eta\end{aligned}$$

for all $t\in(1,\infty)_{\mathbb{R}}$, any $n_k\in\mathbb{N}$. Therefore, every solution of (4.4) is oscillatory, by Corollary 4.2 and Remark 2.1 provided that

$$\min_{k\in\{1,3,\dots,n-1\}}\bigg\{nb_0-\frac{k}{4\beta_0^{k+1}}\bigg\}=nb_0-\frac{n-1}{4\beta_0^n}>0\Rightarrow\beta_0^nb_0>\frac{n-1}{4n}.$$

Notice that (4.3) is true for every $k\in(0,n)_{2\mathbb{N}-1}$ and every sufficiently large $r,s\in[t_0,\infty)_{\mathbb{T}}$ with $\beta_0r>s$.

5 Kamenev-Type Oscillation Criteria for Odd-Order Equations

In this section, we extend our results to odd-order neutral delay dynamic equations.

Theorem 5.1. *Assume that* $n \in 2\mathbb{N}-1$ *and* $A \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[0,1]_{\mathbb{R}})$ *with* $\limsup_{t\to\infty} A(t) < 1$. *Assume also that*

$$(-1)^{n-1}\int_{t_0}^{\infty} h_{n-1}(t_0,\sigma(\eta))B(\eta)\Delta\eta = \infty \tag{5.1}$$

holds, and for every $k \in (0,n)_{2\mathbb{N}}$, *there exist a fixed* $n_k \in \mathbb{N}$ *and a function* $\varphi_k \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+_0)$ *such that* (4.1) *holds for all sufficiently large* $r,s \in [t_0,\infty)_{\mathbb{T}}$ *with* $\beta(r) > s$. *Then, every solution of* (1.1) *oscillates or tends to zero asymptotically.*

Proof. The proof makes use of similar arguments if Kiguradze's theorem holds for $m \in (0,n)_{2\mathbb{N}}$ and for y_x defined by (3.3). And if $m = 0$, then we apply [13, Theorem 3.1] since y_x is bounded, and this implies boundedness of x. Then, we see that x tends to zero asymptotically. □

Theorem 5.2. *Assume that* $n \in 2\mathbb{N}-1$ *and* $A \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[-1,0]_{\mathbb{R}})$ *with* $\liminf_{t\to\infty} A(t) > -1$. *Assume also that* (5.1) *holds, and for every* $k \in (0,n)_{2\mathbb{N}}$, *there exist a fixed* $n_k \in \mathbb{N}$ *and a function* $\varphi_k \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+_0)$ *such that* (4.3) *holds for all sufficiently large* $r,s \in [t_0,\infty)_{\mathbb{T}}$ *with* $\beta(r) > s$. *Then, every solution of* (1.1) *oscillates or tends to zero asymptotically.*

Proof. The proof is similar to that of Theorem 5.1, and hence we omit it. □

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