A Victorian Age Proof of the Four Color Theorem

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Abstract

In this paper we have investigated some old issues concerning four color map problem. We have given a general method for constructing counter-examples to Kempe's proof of the four color theorem and then show that all counterexamples can be rule out by re-constructing special 2-colored two paths decomposition in the form of a double-spiral chain of the maximal planar graph.
In the second part of the paper we have given an algorithmic proof of the four color theorem which is based only on the coloring faces (regions) of a cubic planar maps. Our algorithmic proof has been given in three steps. The first two steps are the maximal mono-chromatic and then maximal dichromatic coloring of the faces in such a way that the resulting uncolored (white) regions of the incomplete two-colored map induce no odd-cycles so that in the (final) third step four coloring of the map has been obtained almost trivially.

1 Introduction

Four color map coloring problem is to color regions of a (normal) map M with at most four colors so that neighbor regions (countries) would have receive different colors. This simple problem posed and conjectured to be true for all maps by Guthrie in 1852 [1],[32]. Its correct proof was first given in 1976 and repeated several times by the same method by the help of a computer [2]-[5]. The author has given two non-computer proofs of the four color theorem based on spiral chains in planar graphs [6],[7],[8].

In this paper we will give another one based on step-wise mono-chromatic coloring, two coloring and then four coloring of any given normal map M, i.e., four coloring of the faces of any cubic planar graph. Therefore our proof suits with the mathematics of the Victorian age [9] in which the four color problem arose. In order to make a smooth transition to the proof we will re-investigate particularly counter-examples ("bad" examples) to Kempe's proof. Michael Rosellini in his undergraduate project summaries existing proofs together with the historical initial efforts. For his study of an counter-example he has chosen the paper of Holroyd and Miller entitled "The example that Heawood should have given" [10] which is actually same example given by Errera [31] but drawn in the plane differently [11]. A close look to that example reveals a property which leads to a general method for constructing a class of counter-examples. On the otherhand we have given a method to re-color vertices of the "bad" maximal graph around the undecided degree five vertex for which Kempe's argument may fail, so that under the resulting four coloring the graph is decomposed into edge disjoint two paths. Furthermore the shape of the paths as seen from the Figure 1 is a double-spiral chain centered at the undecided vertex. Of course any four coloring of G induces edge disjoint two bipartite graphs but not necessarily connected and in the form of a double-spiral. We have also suggest surveys on the early developments of the four color problem by Saaty [12] and Mitchem [13].

The notion of equitable colorability was introduced by Meyer [17]. That is the sizes of color classes differ by at most one. Similarly equitable labeling of graphs introduced by the author in 1990 [18]. However, an earlier work of Hajnal and Szemérdi [19] showed that a graph G with degree $\Delta(G)$ is equitably k-colorable if $k \geq \Delta(G)+1$. In 1973, Meyer formulated the following conjecture:

Conjecture 1 *(Equitable Coloring Conjecture (ECC) [17]). For any connected graph G, other than a complete graph or an odd cycle, $\chi_{=}(G) \leq \Delta(G)$.*

The Equitable k-Coloring Conjecture holds for some classes of graphs, e.g., outerplanar graphs with $\Delta \geq 3$ [20] and planar graphs with $\Delta \geq 13$ [21]. However the four colorings given for bad-examples in Figure 1 are all equitable 4-coloring.

We have the following claim:

Claim. Let G be a maximal planar graph. Then there exits 4-coloring of G for which at least the sizes of three color classes differ by at most one.

2 Bad Examples for Kempe's Argument

After studying all known bad-examples to Kempe's argument one can reach to the conclusion that it is occurred only for specific planar graphs with specific incomplete four-coloring. Gethner et. al. [22],[23] have investigated Kempe's flawed proof of the Four Color Theorem from a computational and historical point of view. Kempe's "proof" gives rise to an algorithmic method of coloring planar graphs that sometimes yields a proper vertex coloring requiring four or fewer colors. They also investigate a recursive version of Kempe's method and a modified version based on the work of I. Kittell [30].

Let G be an maximal planar graph with n vertices.Let T be the triangulation of G. Let $G_1 \in \{P_1, C_1\}$ and $G_2 \in \{P_2, C_2\}$ be two vertex disjoint paths or cycles such that $|G_1| \approx |G_2|$ and $|G_1|+|G_2| = n$ if under such a decomposition of G every triangle t_i has exactly one edge either from G_1 or G_2 then we say triangu-

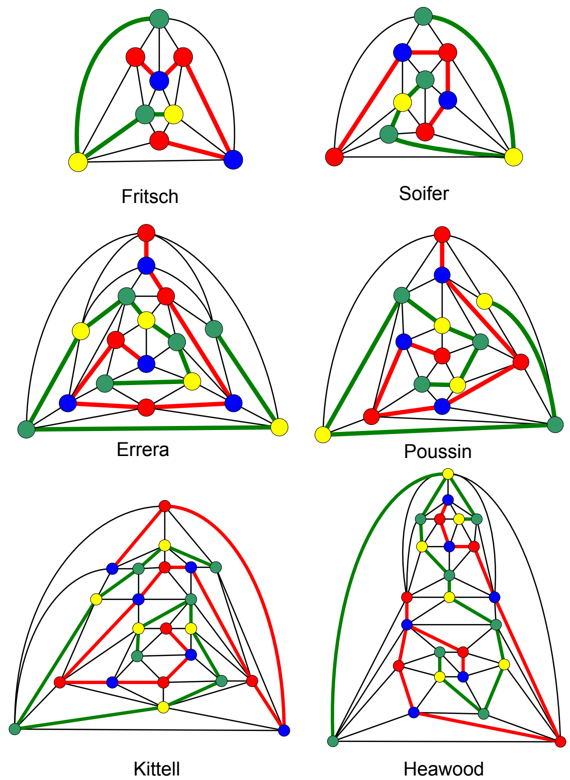


Figure 1: All known counterexamples to Kempe's "proof" with double-spiral chain decompositions.

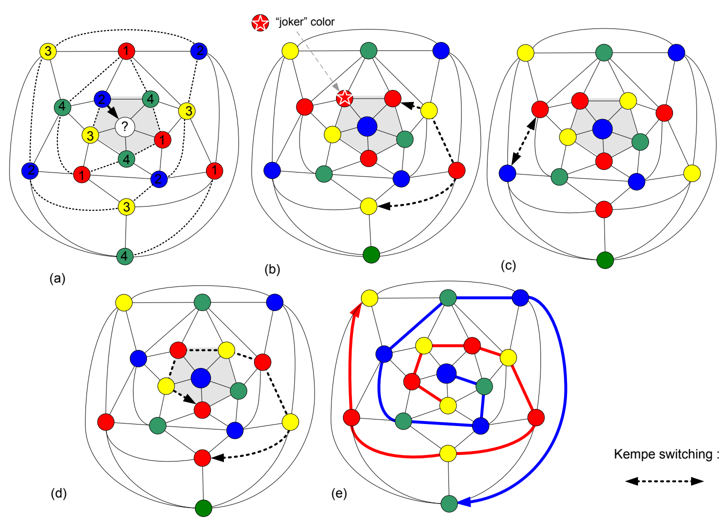


Figure 2: Step-by-step resolution of an impasse in the Errera's graph.

lation T as β-triangulation. If $|C_1| \equiv |C_2| \equiv 0(mod2)$ then 4-coloring of G easily can be obtained. For example in Figure 1 for Fritch's graph $|P_1| = 3, |P_2| = 4$, for Sofier's graph $|P_1| = 4, |P_2| = 3$, for Errera's graph $|P_1| = 8, |P_2| = 7$, for Poussin's graph $|P_1| = 7, |P_2| = 6$, for Kittell's graph $|P_1| = 10, |P_2| = 11$ and finally for Heawood's graph $|P_1^*| = 12, |P_2^*| = 11$, where P_1^*, P_2^* are acyclic graphs. We choose the four colors as $\{\mathbf{R}ed, \mathbf{B}lue, \mathbf{Y}ellow, \mathbf{G}reen\}$ or for another reason $\{\mathbf{B}rown, \mathbf{G}reen, \mathbf{d}ark\mathbf{B}lue, \mathbf{l}ight\mathbf{B}lue\}$ or $\{1, 2, 3, 4\}$. Moreover *white* colored vertex or region in a map means awaiting color from the four-color set.

One of the important property of an "real" bad-example to Kempe's argument is that occurrence of Kempe tangling must be independent from the order of the selection of Kempe-chains. For example Errera's bad example (first incomplete 4-coloring of Figure 2) satisfies this condition. Now consider $C_{5,in} = \{B, G, R, G, Y\}$ that surrounds undecided white vertex. Consider also two disjoint 2-colored cycles of length six (shown dashed lines),i.e., $C_{6,in} = \{R, G, R, G, R, G\}$ and $C_{6,out} = \{B, Y, B, Y, B, Y\}$ which forms an triangulated ring [24]. After cyclically shifting the colors in $C_{6,in}$, insert the Red "joker" color instead of Blue vertex in $C_{5,in} = \{B, G, R, G, Y\}$. Then the three Kempe chain switchings; $Ch(R, Y, R, Y), Ch(R, B)$ and $Ch(R, Y, R, Y, R, Y, R)$ (see Figure 2) resolves the impasse and a double spiral chain results [25].

2.1 Construction of a class of bad-examples

A triangulated ring is a 2-connected planar graph G_r with two faces F_i and F_o whose facial walks are the (induced) cycles C_i and C_o respectively such that: (a) $V(C_i) \cup V(C_o) = V(G)$ and $V(C_i) \cap V(C_o) = \phi$ where indices i and o are being used to denote the inner and outer cycles (faces)of the graph and (b) every face other than F_i and F_o is a triangle. We further assume that all triangles in G_r are of type β-triangle, that is exactly one edge of the triangle belongs C_i or C_o. Since we are interested in small size "bad-example" graphs we consider only $|C_i| = |C_o| = 4, 6$. Let us give a simple lemma first.

Lemma 1. *A triangulated ring G_r with a β-triangulation and with $|C_i| = |C_o| \equiv 0(mod2)$ can be 4-colored such that C_i and C_o colored disjoint 2-color classes.*

Proof. Since the inner and outer cycles are of even length; color inner cycle, say with blue and red and outer cycle with green and yellow. The β-triangulation of G_r prevents any color conflicts in the four coloring.

Now we can construct a maximal planar graph G from G_r as follows: (i) Place an edge e_i inside of the inner face F_i and place also an edge e_o inside of the infinity (finite if the map embedded on sphere) outer-face F_o. (ii) Make a maximal planar graph G by joining the end vertices of e_i with the vertices of F_i and by joining the end vertices of e_o with the vertices of F_o such that resulting triangulation is a β-triangulation and e_o is an outer-edge of G. We say inner-cycle $C_{i,in}$ is a *handcuffs* for the inner-edge e_i. Similarly we say outer-cycle $C_{i,out}$ is a *handcuffs* for the outer-edge e_o. The reason of this terminology will be clearer when we extract bad-examples for Kempe's argument from G. We

will be interested in the following four coloring of G: Color vertices of $C_{i,in}$ and e_o by R and B colors and color vertices of $C_{i,out}$ and e_i by Y and G colors. This four coloring of G is an proper coloring since under the cycle and edge decomposition, the triangulation is a β-triangulation. In case of cycles are of length six, let $C_{6,in} = \{u_1, u_2, ..., u_6\}$, $e_o = \{u_7, u_8\}$ and $C_{6,out} = \{v_1, v_2, ..., v_6\}$, $e_i = \{v_7, v_8\}$. Let us assume that under the β-triangulation of G we also have two special Kempe-chains as follow:
(i) (Y, R)-chain $\Rightarrow ch(v_1, u_2, v_7, u_6, v_5, u_7)$
(ii) (Y, B)-chain $\Rightarrow ch(v_1, u_1, v_7, u_3, v_5, u_8)$
Now we are ready to construct the twin-bad-example graphs for Kempe's argument.

(a) **Twin-graph G_1.** (*Trouble in inner-face*). Delete any two edges, other than the edges of $C_{6,in}$ and e_i, of β-triangulation bounded by $C_{6,in}$ and e_i such that the resulting new face $F_{5,in}$ contains the edge e_i in its boundary cycle of length 5. For example we have deleted edges (v_7u_3) and (v_8u_3) from G and obtain a new cycle (face) $C_{5,in} = (v_7, u_2, u_3, u_4, v_8)$. Now we claim that under the existing four coloring of G if we place a new vertex v_x inside of face $F_{5,in}$ and join all vertices of $C_{5,in}$ to vertex v_x then the resulting incomplete four coloring of the modified planar graph G_1 is an bad-example to Kempe's argument. That is the four colors appear in $C_{5,in} = \{v_7, v_8, u_4, u_3, u_2\}$, (i.e., see Figure 3(b): (Y, G, R, B, R))cannot be reduced to three colors by any Kempe-chain switching. One reason of this impasse is that (Y, G) (resp. (R, B)) end-vertices colored edge e_i (resp. e_o) cannot be extended due to (R, B) (resp. (Y, G) colored handcuffs cycle. Moreover (G, B)-chain $ch(v_8, u_5, u_8, v_2, u_3)$ and (B, Y)-chain $ch(u_3, v_3, u_8, v_1, u_1, v_7)$ would prevent to reduce the number of colors to three on the vertices of $C_{5,in}$. Hence incomplete four coloring of the maximal planar graph G_1 with 17 vertices shown in Figure 3(b) is an bad-example to Kempe's argument.
Note that we have the same decomposition as above if we consider;
(G, R) cycle $C_{6,in} = \{v_6, u_6, v_8, u_4, v_4, u_7\}$ and $e = \{u_2v_2\}$ and
(Y, B) cycle $C_{6,in} = \{v_1, u_1, v_7, u_3, v_3, u_8\}$ and $e = \{u_5v_5\}$.

(b) **Twin-graph G_2.** (*Trouble in outer-face*). The second bad-example graph G_2 can be obtained from G by deleting edges (v_1u_7) and (v_1u_8). Outer-cycle of G_2 is $C_{5,out} = (u_7, v_6, v_1, v_2, u_8)$ that has been colored by R, G, Y, G, B (see Figure 3(b)). Now if we place the new vertex v_x in the outer-face of G_2 and join to the vertices of $C_{5,out}$ then v_x cannot be colored by the use of Kempe's argument.
This due to the (Y, R)- and (Y, B)-chains mentioned in (i) and (ii) before. Moreover switching of colors of the end-vertices of the edges (v_6u_1) or (v_2u_2) would not reduce the number of colors on $C_{5,out}$. Hence G_2 is an bad-example graph to Kempe's argument.

In Figure 4(a) and (b) we have shown another twins bad-example graphs G_1 and G_2 with 13 vertices where the handcuffs cycles $C_{4,in}$ and $C_{4,out}$ are of length four. Moreover comparing the known-bad-example graphs shown in Figure 1

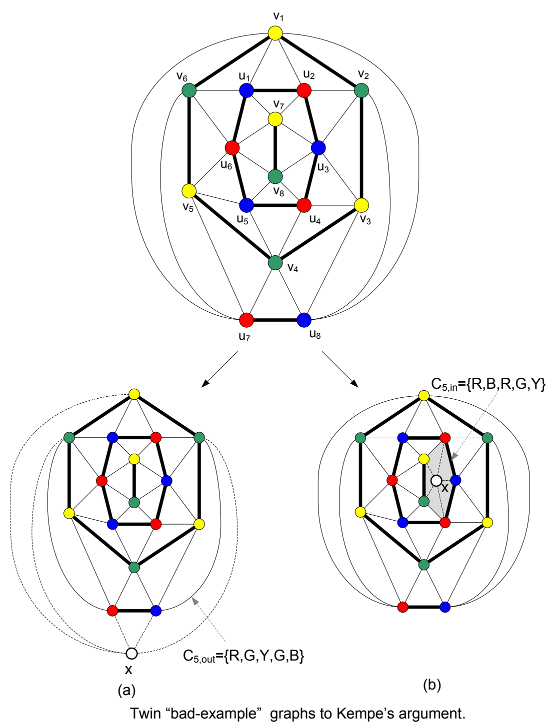


Figure 3: Four coloring of an generator maximal planar graph with β-triangulation: (a) and (b) twin bad-example graphs for the Kempe's argument.

the graphs G_1 and G_2 are the smallest bad-examples in which occurrence of an impasse is not depend on the order of Kempe chain switching.In Figure 4(c) we also have illustrated double-spiral chain four coloring of the bad-example of Figure 4(b). It is not difficult to show that this is possible for all bad-example graphs [24].

In the next section we propose a new proof for the four color theorem without using Kempe-chains based on step-by-step coloring of the faces of cubic planar maps.

3 A New Proof of the Four Color Map Theorem

A more courageous title of this section would be "*How to create a four colored world in three steps?*" It is well-known and without doubt that four color theorem is true. What are the reasons for a lengthy existing proofs by the use of a computer? One answer would be going to the long way which has been forced by the false Kempe's "proof", see for example Birkhoff's reducibility of double C_5 (actually overlapped 4 cycles of length 5),e.g., Birkhoff's diamond

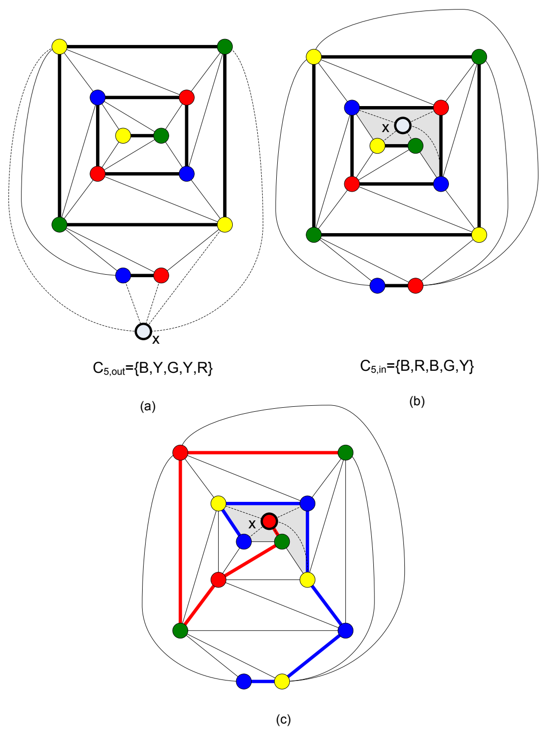
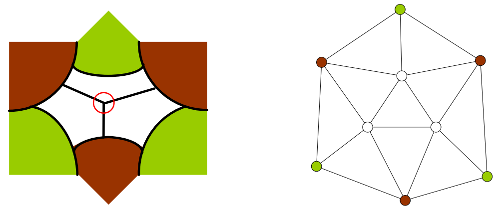
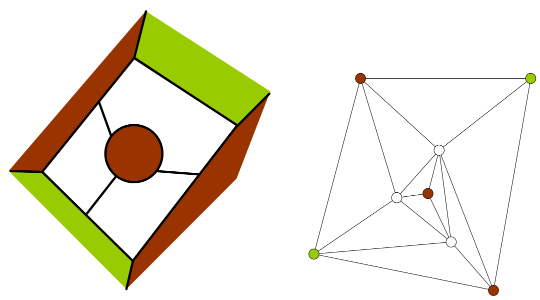


Figure 4: (a),(b)Four coloring of twin bad-example maximal planar graphs with $2C_4 \cup \{e_{in}\} \cup \{e_{out}\}$ and (c) double-spiral chain coloring of (b)

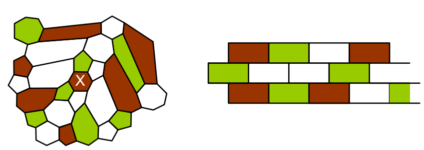


(a)

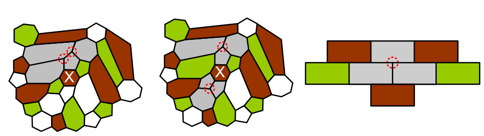


(b)

Figure 5: Bad configurations in a maximal two colored map that require five colors: (a) with unwanted spot (this was the case of a bad-example to Kempe's argument; see 2-color handcuffs cycle C_6), (b) without unwanted spot.



(a)



(b)

Figure 6: Good and bad assignments of low-land green-color regions in a two-colored map.

[14] while cycle of length five C_5 is not reducible. Another answer would be over looking difficulties of the planar three colorability problem in the light of Grotzsch and Heawood's theorems [15],[16]. In this section we will be giving a new proof of the four color map theorem in which we have implicitly by pass the three-coloring problem of planar graphs within the constructive proof.

In fact our algorithmic proof implies the following theorem without relying on the four color theorem [26],[27]:

Theorem 1. *Every planar graph can be decomposed into the edge disjoint union of two bipartite graphs.*

Let us denote by M an normal map with $n+1$ regions, where $(n+1)$th region r_{n+1} is the outer-region of M. M can be equivalently represented by a cubic planar graph $G_c(M) = (V_c, E_c)$, where V_c is the set of vertices associated with the crossing of pairwise three neighbor regions, and E_c is the set of edges in the form of Jordan curve associate with the boarder of two neighbor regions between two vertices. In order to make the map-coloring algorithm more visible and meaningful let us define the four-color set as $C = \{B, G, dB, lB\}$, where
- B denotes *brown* color and when it is assigned on to the *white* background color the corresponding region becomes a "high-land".
- G denotes *green* color and when it is assigned on to the *white* background color the corresponding region becomes a "low-land".
- dB denotes *dark-blue* color and when it is assigned on to the *white* background color the corresponding region becomes a "deap sea".
- lB denotes *light-blue* color and when it is assigned on to the *white* background color the corresponding region becomes a "shallow-sea".

Initially the given map colored all by background color white and at the end of the coloring algorithm (three steps) it will be colored by the colors C and no white color remains on the map. Clearly we will show that this is always possible for any map M.

By $M(B)$ we denote a map in which maximal number of its regions colored by B (mono-chromatic coloring) where the term maximal means that any additional brown region (high-land) results color conflict and all the remaining regions are background-color white. Similarly by $M(B,G)$ we denote a map obtained from $M(B)$ in which maximal number of its white regions colored by G. Hence $M(B,G)$ is an maximal two-coloring of M.

Definition 1. *In a mono-chromatic coloring of map $M(B)$ if an vertex v is not incident to any brown colored region then v is called unwanted-spot or simply a spot . Furthermore if the map $M(B)$ is spot-free then the map $M(B)$ is called clean map.*

Definition 2. *Spiraling of a map M is a process of ordering and labeling the faces (regions), starting from the outer-region r_{n+1} and selecting always outer next region r_i neighbor to the previous region r_{i+1} in the form of a spiral.*

Note that depending on the adjacency of the regions of the map M we may

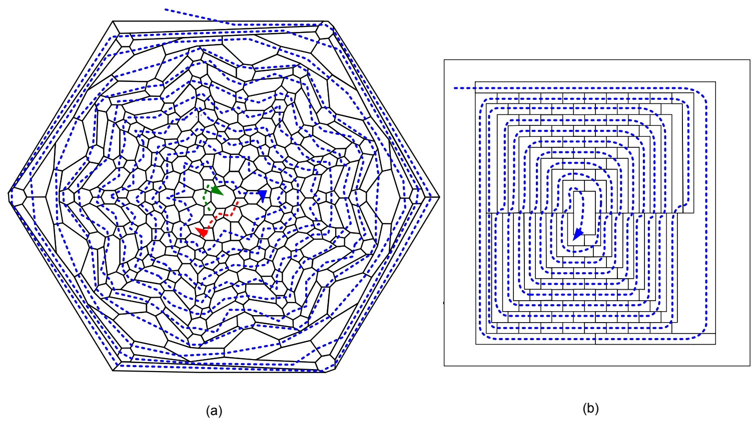


Figure 7: Spiraling of the Haken and Appel's and Martin Gardner's maps.

have several spirals but the ordering of the regions is uniquely determined by the initial region and next one with the direction selected e.g., clockwise or counter clockwise. Similar definition has been given for maximal and cubic planar graphs in [6],[7]. For an illustration spiraling see the nested three spirals shown in blue, red and green colors in Figure 7.

3.1 The Map Coloring Algorithm.

Main feature of the coloring algorithm is the use of each of the four colors one-by-one and preparing the conditions satisfied for the next step.

Step 1. *Maximal mono-chromatic coloring of high-lands map* $M(B)$.
Let $S = \{r_{n+1}, r_n, r_{n-1}, ..., r_1\}$ be the spiral ordering of the faces of map M. Color outer-face r_{n+1} of M with B. Along the spiral S color next white region $r_i \in S$ with B by the following rule:
(i) All the first neighborhood of the region r_i remain in white (uncolored).
(ii) If any white region $r_j, j > i$ is colored, that is $c(r_j) = B$ then a color-conflicts arises.
(iii) At least one of the second neighborhood region of r_i would be colored by B.
Using (i)-(iii) and spiraling S the maximal mono-chromatic set of k regions can be obtained. Let us call the map M after the coloring as $M(B)$. Let us also denote the spots of $M(B)$ with a set $P = \{p_1, p_2, p_3, ..., p_k\}$ where $k < n$. That is P is the set of triply neighbor white regions of the map $M(B)$ where some of the white regions may be overlapped.
The output of the step 1 is simply maximal disjoint of highland islands all colored in brown.
We have also the following simple property of $M(B)$.

Lemma 2. *The spots of the triply neighbor white regions of the map $M(B)$ cannot induces a cycle.*

Proof. Let us assume that a region r colored by B has been surrounded by an cycle of spot vertices. Hence regions in the second neighborhood must be also all white. But (iii) we have colored at least one of the region in the second neighborhood in B and that breaks the cycle of the spots into a path.

As it has been seen that Step 1 is rather straight forward and map $M(B)$ can easily be obtained for any M. Assuming the maximal mono-chromatic coloring of $M(B)$ as a base, it is not such an easy task to obtain dichromatic map $M(B,G)$. In the next step we will give the details and proofs that starting from mono-chromatic $M(B)$ it possible to two-coloring of $M(B,G)$ with a set of properties that satisfies four colorability of the whole map. That is we will show that by assigning color green (color for low-land) to the some of the white regions of $M(B)$ we obtain maximal dichromatic coloring of $M(B,G)$ without any spots, without any even (B,G)-ring and without odd any W-ring (white-rings in $M(B,G)$).

Let us remind the role of two-colored even cycles (handcuffs) in constructing new counter-examples to Kempe's argument in Section 2.

In Figure 5 we have demonstrated one of reason of an bad assignment of color green in $M(B)$. That is even-ring $R(B,G)$ would prevent to complete coloring of white regions with four colors. Another reason of an bad assignment of color green is that, not leaving any room for the other colors to vanish the white-spots (see Figure 6 (a) and (b)).

Lemma 3. *Mono-chromatic (green) spiral-chain coloring of the white regions of the map $M(B)$ results in a spot-free map $M(B,G)$.*

*Proof.*If a spot-vertex remain in $M(B,G)$ it would be one of the bad configurations illustrated in Figures 5 and 6 (b). But this bad configurations can only occur when green color assigned without considering the maximum number of spots of the white region. However this has been protected by Step 2 (i) in the algorithm.

Lemma 4. *Two coloring of the map $M(B,G)$ can be extended to four coloring iff white-regions of $M(B,G)$ induced a (not necessarily connected) bipartite subgraph.*

Proof. Since the white regions in $M(B,G)$ induce a bipartite graphs they can be colored with two colors (dB, lB). Otherwise the maximal two-colored map $M(B,G)$ has an odd cycle formed by all white regions and then we need the fifth color.

Theorem 2. *The map $M(B,G)$ obtained by the Map-Coloring-Algorithm in Step 2 can be extended to a four coloring of M.*

Proof. Proof follows from Lemmas 2,3,4 and 5.

Definition 3. *Let $r_i, r_j, r_k \in M, i \neq j \neq k$. If $v_a, v_b \in r_i, r_j$ and $v_c, v_d \in r_j, r_k, v_a \neq v_b \neq v_c \neq v_d$ then the region r_j is called tunnel connecting r_i and r_k.*

The next lemma is related with the two-colored even-cycles in $M(B,G)$.

Lemma 5. *Let $M(B,G)$ be a two colorable cubic planar map without tunnel regions. Let $M(B,G)$ be surrounded by two rings R_1 and R_2 where R_1 is an odd-ring within the first neighborhood of $M(B,G)$ and R_2 is an even-ring within the second neighborhood of $M(B,G)$. If $M(B,G)$ and $R_2(B,G)$ have been two colored by B and G then the map $M = R_1 \cup R_2(B,G) \cup M(B,G)$ cannot be extended to a four coloring.*

Step 2. *Maximal dichromatic coloring of high-low-lands map $M(B,G)$.*
We use the same spiraling S of the map $M(B)$. While assigning color green G to a white region consider the following two conditions:
(i) While assigning green color to white regions give priority to the white-region which has maximum number of spot vertices in $M(B)$;
(ii) Do not create any (B,G)-ring $R(B,G)$ which contains an inside odd white-ring $R(W)$.

Lemma 6. *The map $M(B,G)$ obtained by the map coloring algorithm step 1 and 2 has no odd-white cycles.*

Proof. We have eliminated all spot vertices in $M(B,G)$ so the length of any odd-white cycle would be ≥ 5. Let us assume that there exists a white-ring (cycle) $R_5(W)$ in $M(B,G)$ of length 5. Let us denote by $M_{in}(B,G)$ the inner two colored map of $R_5(W)$. Let R_{out} be denote the outer ring that surrounds $R_5(W)$. In case one of the region r_i is an tunnel region then there would be a (B,G)-chain breaking the white-ring $R_5(W)$. Hence the outer ring has five regions which colored by G, B, G, B, W. Then the white region must have a spot-vertex common with $R_5(W)$. A contradiction. Then let us assume that R_{out} has six regions (even number). In this case R_{out} should be colored alternatingly by B and G. But by Step 2 (ii) in the algorithm we don't let any even (B,G)-ring around the odd-white ring.

Step 3. Four coloring of $M(B,G,lB,dB)$.
Since maximal dichromatic map $M(B,G)$ has only even white-rings and acyclic white regions, i.e., forest of disjoint trees and paths we can easily color them with light-blue lB and dark-blue dB.

That is at the end of Step 3 the initial all-white normal map M transformed into four colored map of $M(B,G,lB,dB)$ with the regions of high-lands, low-lands, deep-seas and shallow-seas.

From Theorem 2 we re-state the famous four color map theorem.

Theorem 3. *All cubic planar maps are* 4*-colorable.*

3.2 Two well-known maps

The map coloring algorithm has been illustrated by the two well-known maps. Figure captions give the details.

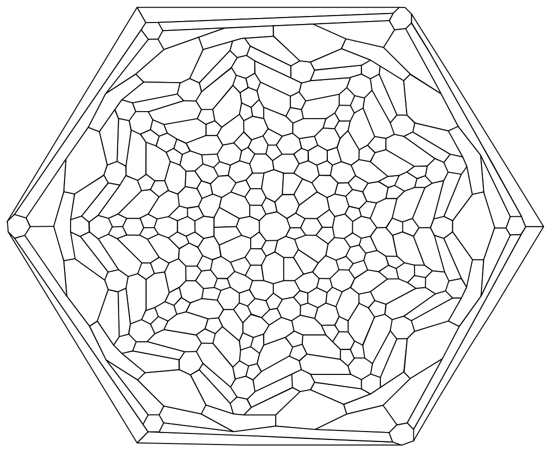


Figure 8: The Haken and Appel's map. This map has been taken from Ed Pegg Jr's mathpuzzle.com/4Dec2001.htm. Haken and Appel needed a computer to 4-color the following hardest-case map, which has been presented in a slightly different form. In this appendix we will explain step-by-step our algorithmic proof of the four color theorem on this map.

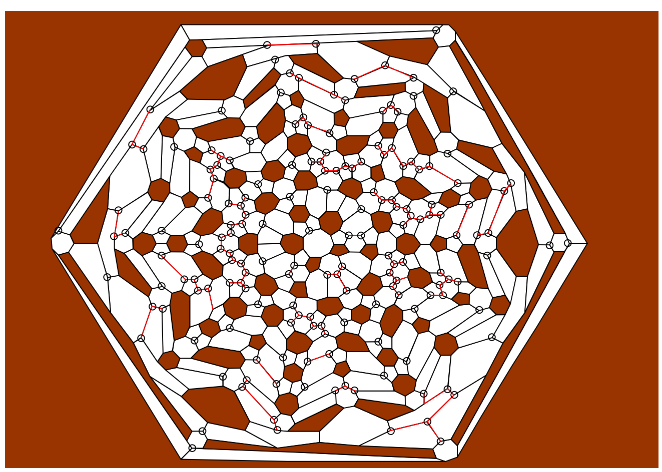


Figure 9: Maximal mono-chromatic coloring of high-land (brown) regions. Note that we start coloring from the outer region and must be all adjacent to white (not colored) regions. Intersection of three adjacent regions have been shown with small circles (unwanted spots) and must be vanished as shown in Figure 10 in the maximal 2-coloring of the map.

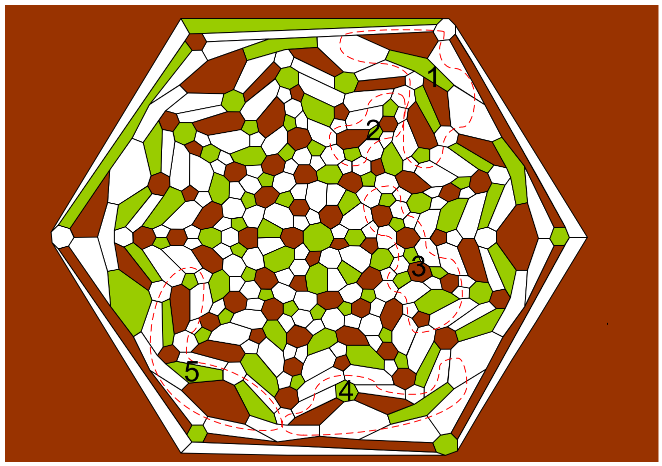


Figure 10: Maximal two coloring of high-land (brown) and low-land (green) regions. Green coloring starts from the upper white region (can be started any white region adjacent to outer region). Trace of green regions form a spiraling in the clockwise direction and at each step at least one "circle" of Figure 9 is vanished by the assignment of the green color to a white region.By red-dashed curves we have shown five even white-rings (even-cycles) around the brown-green (high-lowland)islands. The rest of white regions induce an acyclic graph

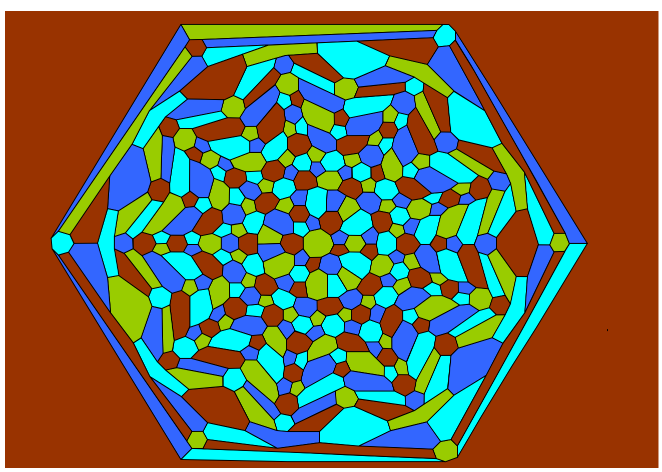


Figure 11: Four coloring of Appel and Haken's map; two coloring of deap sea (dark blue) and shallow sea (light blue) regions of the two colored map of Figure 10. Here two colors is enough for the white regions since the induced dual-graph is bipartite.

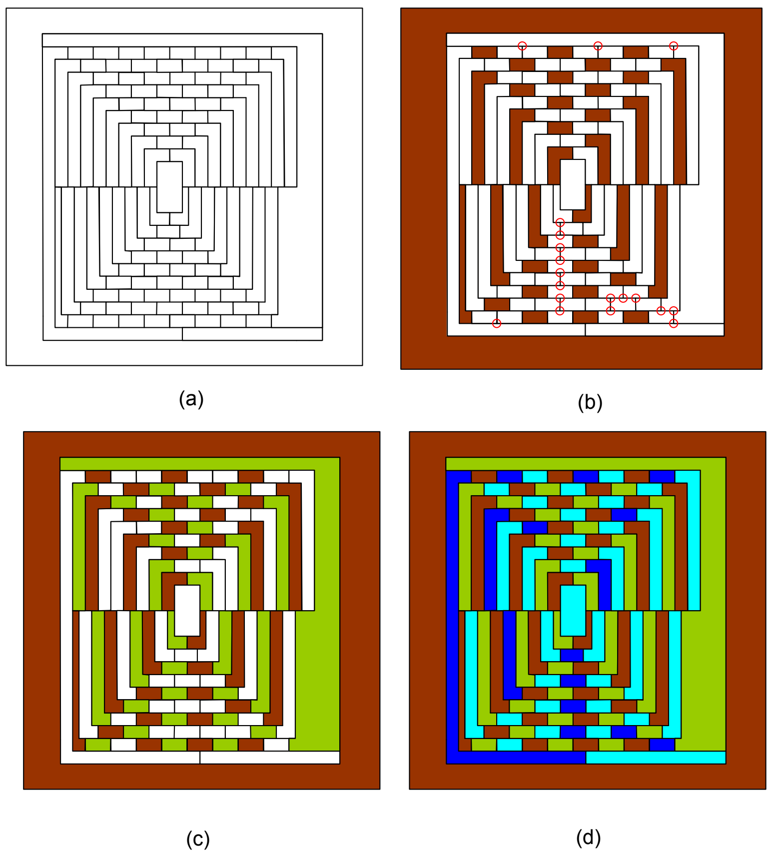


Figure 12: Martin Gardner's April Fool's joke (1975). (a) The "counter-example" map, (b) Brown highland islands, (c) Brown-green high-low islands and (d) The four colored map. Note that (i) each color spiraling in the map and (ii) white regions in (c) induced disjoint union of acyclic subgraphs. Wagon has given four coloring of the April's Fool's map by using Kempe's original algorithm without facing any impasse [28],[29].

4 Concluding remarks

We extract the following from the first page of Appel and Haken's paper [3]:

The first published attempt to prove the Four Color Theorem was made by A.B. Kempe in 1879. Kempe proved that the problem can be restricted to the consideration of "normal planar maps" in which all faces are simply connected polygons, precisely three of which meet at each vertex. For such maps he derived from Euler's formula the equation

$$4p_2 + 3p_3 + 2p_4 + p_5 = \sum_{k=7}^{k_{max}} (k-6)p_k + 12$$

where p_i is the number of polygons with precisely i neighbors and k_{max} is largest value of i which occurs in the map. This equation immediately implies that every maximal planar map contains polygons with fewer then six neighbors. In order to prove the Four Color Theorem by induction on the number p of polygons in the map ($p = \sum p_i$), Kempe assumed that every normal map with $p \leq r$ is four colorable and considered a normal planar map M_{r+1} with $r+1$ polygons. He distinguished the four cases that M_{r+1} contained a polygon P_2 with two neighbors, or a triangle P_3 or a quadrilateral P_4, or a pentagon; at least one of these must apply by the equation.

This beautiful Victorian Age deduction works for $P_i, i = 2, 3, 4$ and unfortunately fails for $i = 5$. I think no mathematician of that period would be able to guess the possible length of a proof in future based on reducibility.
In this paper, by choosing direct proof, that is the opposite direction of the above, we have given an algorithmic proof for the Four Color Theorem which is based on an coloring algorithm and avoiding three-colorability in a maximal two-colorable map. The last word about the proofs given in [6],[7],[8] and including this one that uses spiral chains in the coloring algorithm. Simply enable an efficient coloring algorithm and protect us to fall in a situation similar to Kempe-tangling.

Again Appel and Haken argue strongly that [12],[13]:

...it is very unlikely that one could use their proof technique without the very important aid of a computer to show that a large number of large configurations are reducible. Of course, this does not rule out the possibility of some bright *young* person devising a completely new technique that would give a relatively short proof of the theorem.

This paper does not prove the truth of the first sentence but it does prove that the second sentence is wrong, not only just because of the length of the proof.

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